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Notes on topological BL-algebras

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Abstract

In this study, we aim to solve the open problem posed by Zahiri and Borzooei [23], i.e., given an arbitrary linear topological BL-algebra $(L, \mathcal{U})$, under what condition does its topology $\mathcal{U}$ coincides with the new one $\mathcal{V}$ induced by a system of filters. We investigate this problem for both finite BL-algebras and infinite BL-algebras. In the finite case, we give a necessary and sufficient condition. In the infinite case, we consider the conditions where the original topology $\mathcal{U}$ is zero-dimensional and $(L, \mathcal{V})$ is an l–BL-algebra, respectively. In addition, we modify some examples of topological BL-algebras mentioned by Borzooei et al.

Keywords: Filter; l–BL-algebra; Topological BL-algebra; Zero-dimensional space

1. Introduction

In 1998, Hájek [12] introduced a very general many-valued logic called basic logic (or BL) in order to formalize the many-valued semantics induced by a continuous t-norm on the unit real interval [0, 1]. This BL is actually a common fragment of three important many-valued logics: Łukasiewicz logic, Gödel logic, and product logic. The Lindenbaum–Tarski algebras for BL are called BL-algebras. In addition to their interest in terms of logic, BL-algebras have important algebraic properties and they have studied intensively from an algebraic perspective.

According to the viewpoint of the School of Bourbaki, there are three mother structures in mathematics from which all other mathematical structures can be generated and they are not reducible from one to the other: algebraic structures, topological structures, and order structures. The interaction between topological structures and order structures is a stimulating topic in mathematics and computer science, e.g., in the theory of domains [10] and the theory of locales [14]. Topologies and algebras naturally come in contact in representation theory [15] and topological groups [20]. In recent decades, several researchers have proposed a number of algebraic structures associated with logical systems equipped with topologies (further details may be found in [4–7,9,11,13,17,19,23]). Recently, Zahiri and Borzooei [23] applied a system of filters $\mathcal{F} = \{F_i : i \in \Lambda\}$ on a BL-algebra L to construct a topology $\mathcal{T}_{\mathcal{F}}$. This topology is linear but also a topology such that all operations of L are continuous, i.e., $(L, \mathcal{T}_{\mathcal{F}})$ is a topological BL-algebra.

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More importantly, $\{L/F_i : i \in \Lambda\}$ is an inverse system of BL-algebras, and the inverse limit of $\{L/F_i : i \in \Lambda\}$ denoted by $\varprojlim L/F_i$ is isomorphic to C/C_1 (as a BL-algebra [23, Theorem 4.7]), and thus they concluded that $\varprojlim L/F_i$ (as a subspace topology of $\prod_{i\in\Lambda} L/F_i$) is a complete topological space and $(L, \mathcal{T}_{\mathcal{F}})$ is complete if and only if $L \cong C/C_1$ (for more details of these topological algebras, please refer to [13,19]). As observed in [20, Proposition 1.1.1 (b)] and [1, Proposition 11.7], inverse limits are unique up to isomorphism. It was also observed by [19] that treated completions are natural examples of profinite TFL_{ew}-algebras, and profinite algebras play an important role in the study of various logic systems [3]. Moreover, based on these previous studies, we considered EQ-algebras [21,22] and used a more generalized family of filters to investigate EQ-algebras. In [21], we concluded that these topological EQ-algebras are zero-dimensional, and thus they are Stone iff they are compact and Hausdorff.

Consider a topological BL-algebra $(L, \mathcal{U})$ such that $\mathcal{U}$ is a linear topology on L. Then, a base β for $\mathcal{U}$ exists such that any element $B \in \beta$ containing the top element 1 is a filter. We denote X as the set of all elements of β that contains 1 and Y is the set of all finite intersections of the elements of X. Suppose that $Y = \{F_i : i \in I\}$. We define a relation $\leq$ on I by $i \leq j$ iff $F_j \subseteq F_i$ for any $i, j \in I$. Then, it follows that $(I, \leq)$ is an upward directed set and Y is a system of filters of L. Y can induce a topology $\mathcal{V}$ on L with a base $\{x/F_i : i \in I, x \in L\}$. In [23, Remark 3.5], it was proved that $\mathcal{U}$ is finer than $\mathcal{V}$ and an open problem was posed regarding the conditions where $\mathcal{U}$ and $\mathcal{V}$ coincide. In the present study, we aim to solve this problem. If L is a finite BL-algebra, we give a condition such that the problem holds and we show that the condition is necessary. Borzooei et al. [5,6] used l–BL-algebras to study the metrizability on (semi)topological BL-algebras and the relationship between separation axioms and (semi)topological quotient BL-algebras. In [4, Example 3.8] and [6, Example 2.10 (ii)], a BL-algebra $\mathcal{I} = (I, \min, \max, \odot, \rightarrow, 0, 1)$ (called the product structure) equipped with a topology $\mathcal{T}$ induced by a base $S = \{[a, b] \cap I : a, b \in \mathbb{R}\}$ was used improperly a few times. In fact, we show that the example is trivial because the topology $\mathcal{T}$ is discrete. In the present study, by endowing a non-trivial topology on $\mathcal{I}$, we show that $\mathcal{I}$ with this topology is a l–BL-algebra. In [4, Example 3.8], another BL-algebra $\mathcal{I}' = (I, \min, \max, \odot, \rightarrow, 0, 1)$ (Gödel structure) was used improperly. When endowed with the same topology $\mathcal{T}$ in I, it was proved that $(\mathcal{I}', \mathcal{T})$ is a semitopological BL-algebra but not a topological BL-algebra. In the present study, we show that $(\mathcal{I}', \mathcal{T})$ is a topological BL-algebra. Inspired by this, under a condition that $(L, \mathcal{V})$ is a l–BL-algebra, we give a positive answer to the open problem. Moreover, we prove that $(L, \mathcal{V})$ is a zero-dimensional space and we give a necessary condition for the problem. Under this condition, we also discuss the suitable conditions for the open problem.

2. Preliminaries and the open problem

In this section, we summarize some definitions and results regarding BL-algebras and topologies, as well as reviewing the open problem posed by [23].

Definition 2.1. (Hájek [12]). A *BL-algebra* is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0), which satisfies the following conditions:

(i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with 1 as the greatest element and 0 as the smallest element,
(ii) $(L, \odot, 1)$ is a commutative monoid,
(iii) $x \leq y \rightarrow z$ if and only if $x \odot y \leq z$ for all $x, y, z \in L$,
(iv) $x \wedge y = x \odot (x \rightarrow y)$ for all $x, y \in L$,
(v) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for all $x, y \in L$.

Let L be a BL-algebra and $A, B \subseteq L$. We write $A * B$ for $\{x * y : x \in A, y \in B\}$, and when dealing with singleton sets we simply write $a * B$ and $A * b$ rather than $\{a\} * B$ and $A * \{b\}$, where $* \in \{\wedge, \vee, \odot, \rightarrow\}$.

The following proposition provides some basic properties of BL-algebras.

Proposition 2.2. *(Hájek [12]). The following properties hold for any BL-algebra L: for all $x, y, z \in L$,*

(B_1) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$,
(B_2) $x \leq y$ *if and only if* $x \rightarrow y = 1$,