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Characterizations of endograph metric and Γ-convergence on fuzzy sets ☆

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Abstract

This paper is devoted to the relationships and properties of the endograph metric and the Γ-convergence. The main contents can be divided into three closely related parts. Firstly, on the class of upper semi-continuous fuzzy sets with bounded α-cuts, we find that an endograph metric convergent sequence is exactly a Γ-convergent sequence satisfying the condition that the union of α-cuts of all its elements is a bounded set in $\mathbb{R}^m$ for each $\alpha > 0$. Secondly, based on investigations of level characterizations of fuzzy sets themselves, we present level characterizations (level decomposition properties) of the endograph metric and the Γ-convergence. It is worth mentioning that, using the condition and the level characterizations given above, we discover the fact: the endograph metric and the Γ-convergence are compatible on a large class of general fuzzy sets which do not have any assumptions of normality, convexity or star-shapedness. Its subsets include common particular fuzzy sets such as fuzzy numbers (compact and noncompact), fuzzy star-shaped numbers (compact and noncompact), and general fuzzy star-shaped numbers (compact and noncompact). Thirdly, on the basis of the conclusions presented above, we study various subspaces of the space of upper semi-continuous fuzzy sets with bounded α-cuts equipped with the endograph metric. We present characterizations of total boundedness, relative compactness and compactness in these fuzzy set spaces and clarify relationships among these fuzzy set spaces. It is pointed out that the fuzzy set spaces of noncompact type are exactly the completions of their compact counterparts under the endograph metric.

Keywords: Endograph metric; Γ-convergence; Compactness; Total boundedness; Hausdorff metric; Fell topology

0. Basic notions

Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{Q}$ be the set of all rational numbers, $\mathbb{R}^m$ be the m-dimensional Euclidean space, $K(\mathbb{R}^m)$ be the set of all nonempty compact sets in $\mathbb{R}^m$, $K_C(\mathbb{R}^m)$ be the set of all nonempty compact convex sets in $\mathbb{R}^m$ and $C(\mathbb{R}^m)$ be the set of all nonempty closed sets in $\mathbb{R}^m$.

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A set $K \in K(\mathbb{R}^m)$ is said to be star-shaped relative to a point $x \in K$ if for each $y \in K$, the line $\overline{xy}$ joining x to y is contained in K. The kernel $\ker K$ of K is the set of all points $x \in K$ such that $\overline{xy} \subset K$ for each $y \in K$. The symbol $K_S(\mathbb{R}^m)$ is used to denote all the star-shaped sets in $\mathbb{R}^m$.

Obviously, $K_C(\mathbb{R}^m) \subsetneq K_S(\mathbb{R}^m)$. It can be checked that $\ker K \in K_C(\mathbb{R}^m)$ for all $K \in K_S(\mathbb{R}^m)$.

A fuzzy set u on $\mathbb{R}^m$ is in fact a function u from $\mathbb{R}^m$ to [0,1]. The symbol $F(\mathbb{R}^m)$ is used to represent all fuzzy sets on $\mathbb{R}^m$ (see [8,44] for details). $2^{\mathbb{R}^m} := \{S : S \subseteq \mathbb{R}^m\}$ can be embedded in $F(\mathbb{R}^m)$, as any $S \subset \mathbb{R}^m$ can be seen as its characterization function, i.e. the fuzzy set

$$\widehat{S}(x) = \begin{cases} 1, x \in S, \\ 0, x \notin S. \end{cases}$$

Specially, $\widehat{\emptyset}_m$ represents the fuzzy set on $\mathbb{R}^m$ which is defined by $\widehat{\emptyset}_m(x) \equiv 0$ for any $x \in \mathbb{R}^m$. For simplicity, we denote $\widehat{\emptyset}_m$ by $\widehat{\emptyset}$ if there is no confusion.

For $u \in F(\mathbb{R}^m)$, let $\{u > \alpha\}$ denote the strong α-cut of u, i.e. $\{u > \alpha\} = \{x \in \mathbb{R}^m : u(x) > \alpha\}$, and let $[u]_\alpha$ denote the α-cut of u, i.e.

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R}^m : u(x) \geq \alpha\}, & \alpha \in (0,1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0. \end{cases}$$

For $u \in F(\mathbb{R}^m)$, we suppose that

(i) u is upper semi-continuous;
(ii) u is normal: there exists at least one $x_0 \in \mathbb{R}^m$ with $u(x_0) = 1$;
(iii-1) u is fuzzy convex: $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in \mathbb{R}^m$ and $\lambda \in [0,1]$;
(iii-2) u is fuzzy star-shaped, i.e., there exists $x \in \mathbb{R}^m$ such that u is fuzzy star-shaped with respect to x, namely, $u(\lambda y + (1-\lambda)x) \geq u(y)$ for all $y \in \mathbb{R}^m$ and $\lambda \in [0,1]$;
(iii-3) If $\lambda \in (0,1]$, then there exists $x_\lambda \in [u]_\lambda$ such that $\overline{x_\lambda y} \in [u]_\lambda$ for all $y \in [u]_\lambda$;
(iii-4) $[u]_\alpha$ is a connected set in $\mathbb{R}^m$ for each $\alpha \in (0,1]$;
(iv-1) $[u]_0$ is a bounded set in $\mathbb{R}^m$;
(iv-2) $[u]_\alpha$ is a bounded set in $\mathbb{R}^m$ for each $\alpha \in (0,1]$.

- We use the symbol $F_{USC}(\mathbb{R}^m)$ to denote the set of all fuzzy sets u on $\mathbb{R}^m$ with u satisfying (i).
- We use the symbol $F_{USCG}(\mathbb{R}^m)$ to denote the set of all fuzzy sets u on $\mathbb{R}^m$ with u satisfying (i) and (iv-2).
- We use the symbol $F_{USCB}(\mathbb{R}^m)$ to denote the set of all fuzzy sets u on $\mathbb{R}^m$ with u satisfying (i) and (iv-1).
- We use the symbol $F_{USCGCON}(\mathbb{R}^m)$ to denote the set of all fuzzy sets u on $\mathbb{R}^m$ with u satisfying (i), (iii-4) and (iv-2). So $u \in F_{USCGCON}(\mathbb{R}^m)$ if and only if $u \in F_{USCG}(\mathbb{R}^m)$ and each cut-set of u is a connected set in $\mathbb{R}^m$.

It's easy to see that $F_{USCGCON}(\mathbb{R}^m), F_{USCB} \subsetneq F_{USCG}(\mathbb{R}^m) \subsetneq F_{USC}(\mathbb{R}^m)$.

$F_{USC}(\mathbb{R}^m)$, $F_{USCG}(\mathbb{R}^m)$ and $F_{USCB}(\mathbb{R}^m)$ are said to be **general** fuzzy sets. The other types of fuzzy sets mentioned in this paper are called **particular** fuzzy sets. The α-cuts of general fuzzy sets have **no** assumptions of normality, convexity, starshapedness or even connectedness.

Here we list several common subsets [8,9,35,44] of $F_{USCGCON}(\mathbb{R}^m)$.

- If u satisfies (i), (ii), (iii-1) and (iv-1), then u is a (compact) fuzzy number. The set of all fuzzy numbers is denoted by E^m.
- If u satisfies (i), (ii), (iii-2) and (iv-1), then u is a (compact) fuzzy star-shaped number. The set of all fuzzy star-shaped numbers is denoted by S^m.
- If u satisfies (i), (ii), (iii-3) and (iv-1), then u is a (compact) general fuzzy star-shaped number. The set of all general fuzzy star-shaped numbers is denoted by $\widetilde{S}^m$.

Fuzzy number has been exhaustively studied in both theory and applications [8,23,39–41,44]. $\mathbb{R}^m$ can be embedded in E^m, as any $r \in \mathbb{R}^m$ can be viewed as the fuzzy number

$$r(x) = \begin{cases} 1, x = r, \\ 0, x \neq r. \end{cases}$$

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