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On polygon numbers of circle graphs and distance hereditary graphs

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ABSTRACT

Circle graphs are intersection graphs of chords in a circle and k-polygon graphs are intersection graphs of chords in a convex k-sided polygon where each chord has its endpoints on distinct sides. The k-polygon graphs, for $k \geq 2$, form an infinite chain of graph classes, each of which contains the class of permutation graphs. The union of all of those graph classes is the class of circle graphs. The polygon number $\psi(G)$ of a circle graph G is the minimum k such that G is a k-polygon graph. Given a circle graph G and an integer k, determining whether $\psi(G) \leq k$ is NP-complete, while the problem is solvable in polynomial time for fixed k.

In this paper, we show that $\psi(G)$ is always at least as large as the asteroidal number of G, and equal to the asteroidal number of G when G is a connected distance hereditary graph that is not a clique. This implies that the classes of distance hereditary permutation graphs and distance hereditary AT-free graphs are the same, and we give a forbidden subgraph characterization of that class. We also establish the following upper bounds: $\psi(G)$ is at most the clique cover number of G if G is not a clique, at most 1 plus the independence number of G, and at most $\lceil n/2 \rceil$ where $n \geq 3$ is the number of vertices of G. Our results lead to linear time algorithms for finding the minimum number of corners that must be added to a given circle representation to produce a polygon representation, and for finding the asteroidal number of a distance hereditary graph, both of which are improvements over previous algorithms for those problems.

1. Introduction and preliminaries

The dimension of comparability graphs and the treewidth of graphs are widely studied graph parameters that are important from both algorithmic and structural points of view [29,24]. In this paper, we study an analogous parameter of circle graphs, namely, the polygon number. The three parameters have similar algorithmic and complexity properties, and each of them may be seen as a parameter of an associated representation: a realizer of a partially ordered set, a tree decomposition of a graph, or a polygon representation of a circle graph. Further similarities between the polygon number of a circle graph and the dimension of a comparability graph will be mentioned later.

The k-polygon graphs, for $k \geq 2$, form an infinite chain of graph classes, each of which contains the class of permutation graphs, and the union of which is the class of circle graphs. The polygon number of a given circle graph is the minimum value of k such that the graph is a k-polygon graph. Given a circle graph G and an integer k, determining whether the polygon number of G is at most k is NP-complete, while the problem is solvable in polynomial time for fixed k [12]. Several problems

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that are known to be NP-hard on circle graphs admit polynomial time algorithms for k-polygon graphs when k is fixed, including domination and independence problems, the topological via minimization problem in circuit design [11], and vertex colouring with a fixed number of colours [31]. In addition, small collective additive tree spanners can be constructed efficiently for k-polygon graphs when k is fixed [10], and the bandwidth of a k-polygon graph can be approximated to within a factor of $2k^2$ in polynomial time [26]. The running times of several of these algorithms are of the form $O(f(|V|) \cdot |V|^{g(k)})$ where V is the vertex set of the input graph and f and g are polynomial functions.

Although the polygon number has been a key parameter in algorithm design, and the complexity of computing it is known, little is known about its other properties. In this paper we explore how the polygon number of a circle graph relates to established graph parameters. This gives some insight into how the k-polygon graph classes increase in complexity as k increases, and provides estimates on the running times and approximation ratios of algorithms for k-polygon graphs. Specifically, we show that the polygon number is at least as large as the asteroidal number, with equality for connected distance hereditary graphs other than cliques. This implies that the classes of distance hereditary permutation graphs and distance hereditary AT-free graphs are the same, and leads to a forbidden subgraph characterization of that class. We then show that the polygon number of a circle graph is at most the clique cover number (if the graph is not a clique), at most 1 plus the independence number, and at most $\lceil n/2 \rceil$ where $n \geq 3$ is the number of vertices of the graph. These results give rise to linear time algorithms for computing the minimum number of corners that must be added to a given circle representation to construct a polygon representation, and for computing the asteroidal number of a distance hereditary graph.

We begin with terminology and preliminaries, first for graphs and then for intersection representations of circle and k-polygon graphs. Additional definitions and notation are introduced as needed. For terms not defined here, the reader is referred to [18].

The graphs that we consider are finite and simple, and undirected unless stated otherwise. When the vertex and edge sets of a graph G are not explicitly named, we refer to them as $V(G)$ and $E(G)$, respectively. Let $G = (V, E)$ be a graph. The subgraph of G induced by $W \subseteq V$ is denoted $G[W]$. For $v \in V$, the *neighbourhood* of v is $N_G(v) = \{w \mid vw \in E\}$, the *closed neighbourhood* of v is $N_G[v] = N(v) \cup \{v\}$, and the *degree* of v is denoted $d_G(v)$. The neighbourhood of a subset W of V is $N_G(W) = \{v \in V - W \mid wv \in E \text{ for some } w \in W\}$. The subscript G may be omitted when the context is clear. We use $G - V'$ and $G - E'$ as shorthand for the subgraph of G induced by $V - V'$, and the graph $(V, E - E')$, respectively. A vertex of degree one is called a *leaf*. The chordless cycle on n vertices and the clique on n vertices are denoted by C_n and K_n respectively. The size of a maximum independent set is denoted $\alpha(G)$ and the size of a minimum clique cover is denoted $\kappa(G)$. Using the notation of [25], a set $A \subseteq V$ is called an *asteroidal set* if for every vertex $a \in A$, there is a path between each pair of vertices $x, y \in A - \{a\}$ in $G - N[a]$. The *asteroidal number* of G, denoted $an(G)$, is the cardinality of a maximum asteroidal set of G. An *asteroidal triple* (or *AT*) is an asteroidal set of size three.

A graph is called *AT-free* if it has no asteroidal triple. A graph is a *comparability graph* if its edges can be transitively oriented, and a *cocomparability graph* if it is the complement of a comparability graph. A graph G is a *distance hereditary graph* if, for every connected induced subgraph H of G, the distance between each pair of vertices of H is the same in H as it is in G. We refer the reader to [4] for more information about these graph classes.

The *intersection graph* of a finite collection of sets is the graph containing one vertex for each set, such that two vertices are joined by an edge if and only if the intersection of the corresponding sets is not empty.

A graph is a *circle graph* if it is the intersection graph of a set of chords of a circle. For $k \geq 3$, a graph is a *k-polygon graph* if it is the intersection graph of chords inside a convex polygon with k sides such that each chord has its endpoints on two distinct sides of the polygon. For example, for any $k \geq 3$, C_{2k} is a k-polygon graph and not a $(k-1)$-polygon graph [12]. A graph G where $V(G) = \{v_1, \ldots, v_n\}$ is a permutation graph if there exists a permutation π of $\{1, 2, \ldots, n\}$ such that $v_i v_j \in E(G)$ if and only if $(i - j) \cdot (\pi_i^{-1} - \pi_j^{-1}) < 0$. Equivalently, permutation graphs are the intersection graphs of straight line segments connecting two parallel lines. For reasons that will be made evident below, permutation graphs are considered to be 2-polygon graphs. Therefore:

$$\text{permutation graphs} \equiv \text{2-polygon graphs} \subset \text{3-polygon graphs} \subset \ldots \subset \bigcup_{k=2}^{\infty} k\text{-polygon graphs} \equiv \text{circle graphs.}$$

The *polygon number* of a circle graph G, denoted by $\psi(G)$, is the minimum value of k such that G is a k-polygon graph. In [12], Elmallah and Stewart showed that the problem of determining if $\psi(G) \leq k$ for a given circle graph G and an integer k is NP-complete, and they gave a polynomial time algorithm for solving the problem when k is a fixed integer. They also showed that for a circle graph G with connected components $G_1, G_2, \ldots, G_r$,

$$\psi(G) = \left(\sum_{i=1}^{r} \psi(G_i) \right) - 2(r - 1).$$

As this allows us to determine $\psi(G)$ based on the polygon numbers of the connected components of G, we focus on identifying the polygon number of connected graphs in the analysis below.

A set of chords of a circle is called a *circle representation* for graph G if G is the intersection graph of that set of chords. For example, Fig. 1(a) shows a circle representation for C_5 where each chord c_i has endpoints at points labelled as e_i and e_i'. Two distinct points p_0 and p_1 divide the circle into the two arcs: (p_0, p_1), the open arc that is traced in a clockwise traversal of the circle beginning at p_0 and ending at p_1, and (p_1, p_0) which is defined analogously. For chord c with endpoints at points