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Parameterized complexity of distance labeling and uniform channel assignment problems☆

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ABSTRACT

We rephrase the Distance labeling problem as a specific uniform variant of the Channel Assignment problem and show that the latter one is fixed parameter tractable when parameterized by the neighborhood diversity together with the largest weight. Consequently, the Distance labeling problem is FPT when parameterized by the neighborhood diversity, the maximum p_i and k. This is indeed a more general answer to an open question of Fiala et al.: *Parameterized complexity of coloring problems: Treewidth versus vertex cover.*

Finally, we show that the uniform variant of the Channel Assignment problem becomes NP-complete when generalized to graphs of bounded clique width.

1. Introduction

The frequency assignment problem in wireless networks yields an abundance of various mathematical models and related problems. We study a group of such discrete optimization problems in terms of parameterized computational complexity, which is one of the central paradigms of contemporary theoretical computer science. We study parameterization of the problems by *clique width* and by *neighborhood diversity* (nd), a graph parameter lying between clique width and the size of a minimum vertex cover.

All these problems are NP-hard even for constant clique width, including the uniform variant, as we show in this paper. On the other hand, we prove that they are in FPT with respect to nd. Such fixed parameter tractability has so far only been known only for the special case of $L(p, 1)$ labeling when parameterized by vertex cover [8].

1.1. Distance constrained labelings

Given a k-tuple of positive integers $p_1, \dots, p_k$, called *distance constraints*, an $L(p_1, \dots, p_k)$-labeling of a graph is an assignment l of integer labels to the vertices of the graph satisfying the following condition: Whenever vertices u and v are at distance i, the assigned labels differ by at least p_i. Formally, $\mathrm{dist}(u, v) = i \implies |l(u) - l(v)| \ge p_i$ for all $u, v : \mathrm{dist}(u, v) \le k$. Often only non-increasing sequences of distance constraints are considered. Note that for our results this assumption is not necessary.

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Any $L(1)$-labeling is a graph coloring and vice-versa. Analogously, any coloring of the kth distance power of a graph is an $L(1, \ldots, 1)$-labeling. The concept of $L(2, 1)$-labeling is attributed to Roberts by Griggs and Yeh [14]. The *span* of a labeling is the maximal difference between the labels used. It is denoted by λ. Through our paper we assume that all labels of a labeling of span λ belong to the interval $[0, \lambda]$. It is not difficult to show that whenever l is an optimal $L(p_1, \ldots, p_k)$-labeling, then its span λ is a linear combination of $p_1, \ldots, p_k$ [14,17].

In particular, a graph G allows an $L(p_1, \ldots, p_k)$-labeling of span λ if and only if it has an $L(cp_1, \ldots, cp_k)$-labeling of span $c\lambda$ for any positive integer c.

For computational complexity purposes, we define the following decision problem:

Problem 1 (*Distance Labeling*)**.**

Input:	Graph G, positive integers $\lambda, k, p_1, \ldots, p_k$.
Query:	Is there an $L(p_1, \ldots, p_k)$ labeling of G using labels from the interval $[0, \lambda]$?

When $k, p_1, \ldots, p_k$ are fixed, we speak about the $L(p_1, \ldots, p_k)$-LABELING problem.

The $L(2, 1)$-LABELING problem was shown to be NP-complete by Griggs and Yeh [14] by a reduction from HAMILTONIAN CYCLE (with $\lambda = |V_G|$). Fiala, Kratochvíl and Kloks [9] showed that $L(2, 1)$-LABELING remains NP-complete also for all fixed $\lambda \geq 4$, while for $\lambda \leq 3$ it is solvable in linear time.

Griggs and Yeh conjectured that $L(2, 1)$-LABELING is NP-complete on trees [14], but Chang and Kuo [2] assembled a dynamic programming algorithm for this problem, as well as for all $L(p_1, p_2)$-labelings where p_2 divides p_1. All the remaining cases of the $L(p_1, p_2)$-LABELING problem on trees have been shown to be NP-complete by Fiala, Golovach and Kratochvíl [7]. The same authors showed that $L(2, 1)$-LABELING is already NP-complete on series–parallel graphs [6], which have of tree width at most 2. Note that these results imply NP-hardness of $L(3, 2)$-LABELING on graphs of clique width at most 3 and of $L(2, 1)$-LABELING for clique width at most 6 [3].

On the other hand, when λ is fixed, then the existence of an $L(p_1, \ldots, p_k)$-labeling of G can be expressed in MSO_1, hence it allows a linear time algorithm on any graph of bounded clique width [16].

Fiala et al. [8] showed that for $k = 2, p_2 = 1$ the DISTANCE LABELING problem (i.e. the existence of an $L(p, 1)$-labeling) is FPT when parameterized by p together with the size of the vertex cover. They also ask for the complexity characterization of the related CHANNEL ASSIGNMENT problem. We extend their work to the broader class of graphs and, consequently, in our Theorem 9 we provide a solution for their open problem.

1.2. Channel assignment

Channel assignment is a concept closely related to distance constrained graph labeling. Here, every edge has a prescribed weight $w(e)$ and it is required that the labels of adjacent vertices differ at least by the weight of the corresponding edge. The associated decision problem is defined as follows:

Problem 2 (*Channel Assignment*)**.**

Input:	Graph G, a positive integer λ, edge weights $w : E_G \to \mathbb{N}$.
Query:	Is there a labeling l of the vertices of G by integers from $[0, \lambda]$ such that $\lvert l(u) - l(v)\rvert \geq w(u, v)$ for all $(u, v) \in E_G$?

The maximal edge weight is an obvious necessary lower bound for the span of any labeling. Observe that for any bipartite graph, in particular also for all trees, it is also an upper bound—a labeling that assigns 0 to one class of the bipartition and $w_{\max} = \max\{w(e), e \in E_G\}$ to the other class satisfies all edge constraints. McDiarmid and Reed [20] showed that it is NP-complete to decide whether a graph of tree width 3 allows a channel assignment of given span λ. This NP-hardness hence applies on graphs of clique width at most 12 [3]. It is worth noting that for graphs of tree width 2, i.e. for subgraphs of series–parallel graphs, the complexity characterization of CHANNEL ASSIGNMENT is still open. Only a few partial results are known [21], among others that CHANNEL ASSIGNMENT is polynomially solvable on graphs of bounded tree width if the span λ is bounded by a constant.

Any instance G, λ of the DISTANCE LABELING problem can straightforwardly be reduced to an instance G^k, λ, w of the CHANNEL ASSIGNMENT problem. Here, G^k arises from G by connecting all pairs of vertices that are in G at distance at most k, and for the edges of G^k we let $w(u, v) = p_i$ whenever $\mathrm{dist}_G(u, v) = i$.

The resulting instances of CHANNEL ASSIGNMENT have by the construction some special properties. We explore and generalize these to obtain a uniform variant of the CHANNEL ASSIGNMENT problem.

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