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Parameterized extension complexity of independent set and related problems

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ABSTRACT

Let G be a graph on n vertices and $\mathrm{STAB}_k(G)$ be the convex hull of characteristic vectors of its independent sets of size at most k. We study extension complexity of $\mathrm{STAB}_k(G)$ with respect to a fixed parameter k (analogously to, e.g., parameterized computational complexity of problems). We show that for graphs G from a class of bounded expansion it holds that $\mathrm{xc}(\mathrm{STAB}_k(G)) \leqslant \mathcal{O}(f(k) \cdot n)$ where the function f depends only on the class. This result can be extended in a simple way to a wide range of similarly defined graph polytopes. In case of general graphs we show that there is *no function* f such that, for all values of the parameter k and for all graphs on n vertices, the extension complexity of $\mathrm{STAB}_k(G)$ is at most $f(k) \cdot n^{\mathcal{O}(1)}$. While such results are not surprising since it is known that optimizing over $\mathrm{STAB}_k(G)$ is *FPT* for graphs of bounded expansion and $W[1]$-hard in general, they are also not trivial and in both cases stronger than the corresponding computational complexity results.

1. Introduction

Polyhedral (aka LP) formulations of combinatorial problems belong to the basic toolbox of combinatorial optimization. In a nutshell, a set of feasible solutions of some problem is suitably encoded by a set of vectors, whose convex hull forms a polytope over which one can then optimize using established tools. A polytope Q is said to be an *extended formulation* or *extension* of a polytope P if P is a projection of Q. Measuring the size of a polytope by the minimum number of inequalities required to describe it, one can define the extension complexity of a polytope to be the size of the smallest extension of the polytope. This notion has a rich history in combinatorial optimization where by adding extra variables one can sometimes obtain significantly smaller polytopes. For some recent survey on extended formulations in the context of combinatorial optimization and integer programming see [8,13,20,21].

Since linear (or indeed convex) optimization of a polytope P can instead be indirectly done by optimizing over an extended formulation of P, this concept provides a powerful model for solving many combinatorial problems. Various Linear Program (LP) solvers exist today that perform quite well in practice and it is desirable if a problem can be modeled as a small-sized polytope over which one can use an existing LP solver for linear optimization. However, in recent years super-polynomial lower bound on the extension complexity of polytopes associated with many combinatorial problems have been

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established. These bounds have been generalized to various settings, such as convex extended formulations, approximation algorithms, etc. These results are too numerous for a comprehensive listing, but we refer the interested readers to some of the landmark papers in this regard [4,7,10,19].

Many of the recent lower bounds on the extension complexity of various combinatorial polytopes mimic the computational complexity of the underlying problem. For example, it is known that the *extension complexities of polytopes* related to various NP-hard problems are super-polynomial [1,4,10,18]. One satisfying feature of these lower bounds is that they are independent of traditional complexity-theoretic assumptions such as $P \neq NP$. Though, there also exist polytopes corresponding to polynomial time solvable optimization problems whose extension complexity is super-polynomial. In particular, the perfect matching polytope was shown to have super-polynomial extension complexity by Rothvoß [19]. Hence even if the extension complexity of a problem mimics its computational complexity, lower and upper bounds on the former do *not* follow from the corresponding computational complexity bounds and constitute nontrivial new results of independent interest.

One can naturally ask the related questions in the realm of *parameterized complexity theory*. In this rapidly grown field each problem instance comes additionally equipped with an integer parameter, and the "efficient" class denoted by FPT (*fixed-parameter tractable*) is the one of problems solvable, for every fixed value of the parameter, in polynomial time of degree independent of the parameter. See Section 2 for details.

Similarly as parameterized complexity provides a finer resolution of algorithmic tractability of problems, parameterized extension complexity can provide a finer resolution of extension complexities of polytopes of the problems. We similarly say that a polytope has an *FPT extension* if it has an extension which is, for every fixed value of the parameter, of polynomial size with degree independent of the parameter. Again, see Section 2 for details.

We follow this direction of research with a case study of the *independent-set polytope* of a graph, naturally parameterized by the solution size. We confirm that the extension complexity of the independent-set polytope indeed mimics the parameterized computational complexity of the underlying independent set problem—a finding which is again not implied by the parameterized complexity status of this problem and which is actually a lot stronger than previous related complexity knowledge. Precisely, we prove:

- that the independent-set polytope cannot have an FPT extension for all graphs, independently of any computational-complexity assumptions (Section 3), but
- linear-sized FPT extensions of the independent-set polytope do exist on every graph class of bounded expansion (Section 4).

Seeing the latter result, one may naturally think whether analogous results hold for other similar problems. For example, one may consider the polytope of (induced) subgraphs isomorphic to a given graph F, parameterized by the size of F. Or, more generally, polytopes defined by solutions of non-local problems, such as the polytope of dominating sets of a certain size. While ad-hoc adaptations of our technique to such problems are surely possible, we prefer to give a "metatheorem"—a generic solution aimed at all problems defined in a certain framework.

Namely, we further formulate and prove the following generalizations:

- there is a natural way to assign a definition of a polytope to every graph problem expressible in FO logic, and these polytopes have linear-sized FPT extensions on every graph class of bounded expansion when parameterized by the size of the formula expressing the problem (Section 5),
- for a restricted fragment of FO graph problems, near-linear-sized FPT extensions of the polytopes exist even on so called nowhere dense graph classes (Section 6).

We conclude the paper with some further thoughts and suggestions in Section 7.

2. Preliminaries

We follow standard terminology of graph theory and consider finite simple undirected graphs. We refer to the vertex and edge sets of a graph G as $V(G)$ and $E(G)$, respectively. An *independent set* X of vertices of a graph is such that no two elements of X are adjacent. By a *cut* in a graph G we mean an edge cut, that is, an inclusion-wise minimal set of edges $C \subseteq E(G)$ such that $G \setminus C$ has more connected components than G.

For fundamental concepts of parameterized complexity we refer the readers, e.g., to the monograph [9]. Here we just very briefly recall the needed notions. Considering a problem $\mathcal{P}$ with input of the form $(x, k) \in \Sigma^* \times \mathbb{N}$ (where k is a *parameter*), we say that $\mathcal{A}$ is *fixed-parameter tractable* (shortly FPT) if there is an algorithm solving $\mathcal{A}$ in time $f(k) \cdot n^{\mathcal{O}(1)}$ where f is an arbitrary computable function. In the (parameterized) *k-independent set problem* the input is (G, k) where G is a graph and $k \in \mathbb{N}$, and the question is whether G has an independent set of size at least k.

There is no known FPT algorithm for the k-independent set problem in general and, in fact, the theory of parameterized complexity [9] defines complexity classes $W[t]$, $t \geq 1$, such that the k-independent set problem is complete for $W[1]$. Problems that are $W[1]$-hard do not admit an FPT algorithm unless the Exponential Time Hypothesis fails.

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