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Exact algorithms for weak Roman domination[☆,☆☆]

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ABSTRACT

We consider the Weak Roman Domination problem. Given an undirected graph $G = (V, E)$, the aim is to find a *weak Roman domination* function (wrd-function for short) of minimum cost, *i.e.* a function $f : V \to \{0, 1, 2\}$ such that every vertex $v \in V$ is *defended* (*i.e.* there exists a neighbor u of v, possibly $u = v$, such that $f(u) \geqslant 1$) and for every vertex $v \in V$ with $f(v) = 0$ there exists a neighbor u of v such that $f(u) \geqslant 1$ and the function $f_{u\to v}$ defined by $f_{u\to v}(v) = 1, f_{u\to v}(u) = f(u) - 1$ and $f_{u\to v}(x) = f(x)$ otherwise does not contain any undefended vertex. The *cost* of a wrd-function f is defined by $cost(f) = \sum_{v\in V} f(v)$. The trivial enumeration algorithm runs in time $\mathcal{O}^*(3^n)$ and polynomial space and is the best one known for the problem so far. We are breaking the trivial enumeration barrier by providing two faster algorithms: we first prove that the problem can be solved in $\mathcal{O}^*(2^n)$ time needing *exponential space*, and then describe an $\mathcal{O}^*(2.2279^n)$ algorithm using *polynomial space*. Our results rely on structural properties of a wrd-function, as well as on the best polynomial space algorithm for the Red-Blue Dominating Set problem. Moreover we show that the problem can be solved in linear-time on interval graphs.

1. Introduction

In this paper we investigate a domination-like problem from the exact exponential algorithms viewpoint. In the classical Dominating Set problem, one is given an undirected graph $G = (V, E)$, and asked to find a dominating set S, *i.e.* every vertex $v \in V$ either belongs to S or has a neighbor in S, of minimum size. The Dominating Set problem ranges among one of the most famous *NP*-complete covering problems [9], and has received a lot of attention during the last decades. In particular, the trivial enumeration algorithm of runtime $\mathcal{O}^*(2^n)$[1] has been improved by a sequence of papers [8,16,27]. The currently best known algorithms for the problem run in time $\mathcal{O}^*(1.4864^n)$ using polynomial space, and in time $\mathcal{O}^*(1.4689^n)$ needing exponential space [16].

Many variants of the Dominating Set problem have been introduced and studied extensively both from structural and algorithmic viewpoints. The number of papers on domination in graphs and its variants is in the thousands, and several

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1 The notation $\mathcal{O}^*(f(n))$ suppresses polynomial factors.

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well-known surveys and books are dedicated to the topic (see, *e.g.* , [13]). One of those variants called Roman Domination was introduced in [6], motivated by the articles by Stewart [25] and by ReVelle and K.E. Rosing [23]. In general, the aim is to protect a set of locations (vertices of a graph) by using a smallest possible amount of legions (to be placed on those vertices).

Since then, numerous articles have been published around this problem, which has been studied from combinatorics bounds on the minimum cost or related parameters, characterizations of graphs achieving such bounds and algorithms to compute function of minimum costs (see, *e.g.* , [1,3,5,7,19,20,28]). In particular, this *NP*-complete problem has been tackled using exact exponential algorithms. The first non-trivial one achieved had running time $\mathcal{O}^*(1.6183^n)$ and used polynomial space [17]. This result has recently been improved to $\mathcal{O}^*(1.5673^n)$ [26], which can be lowered to $\mathcal{O}^*(1.5014^n)$ at the cost of exponential space [26]. Moreover, the Roman Domination problem can be related to several other variants of *defense-like* domination, such as *secure domination* (see, *e.g.* , [4,5,12]), or *eternal domination* (see, *e.g.* , [10,11]).

We focus our attention on yet another variant of the Roman Domination problem. In 2003, Henning et al. [14] considered the following idea: location t can also be protected if one of its neighbors possesses one legion that can be moved to t in such a way that the whole collection of locations (set of vertices) remains protected. This variation adds some kind of dynamics to the problem and gives rise to the Weak Roman Domination problem. Formally, it can be defined as follows:

> Weak Roman Domination:
> **Input**: An undirected graph $G = (V, E)$.
> **Output**: A weak Roman domination function f of G of minimum cost.

A weak Roman domination (wrd-function) is a function $f : V \to \{0, 1, 2\}$ such that every vertex $v \in V$ is *defended* (*i.e.* there exists a neighbor u of v, possibly $u = v$, such that $f(u) \geqslant 1$) and for every vertex $v \in V$ with $f(v) = 0$ there exists a neighbor u of v such that $f(u) \geqslant 1$ and the function $f_{u\to v}$ defined by $f_{u\to v}(v) = 1, f_{u\to v}(u) = f(u) - 1$ and $f_{u\to v}(x) = f(x)$ otherwise does not contain any undefended vertex. I.e., one can move one legion from u to v without creating any undefended vertex. The *cost* of a wrd-function f is defined by $cost(f) = \sum_{v\in V} f(v)$.

While several structural results on Weak Roman Domination are known, see, *e.g.* , [4,5,14,22], few algorithmic results are known. Note that a preliminary version of this paper appears in Chapelle et al. (2013) [2]. The problem has been shown to be NP-hard, even when restricted to bipartite or chordal graphs [14] and a linear-time algorithm is known for block graphs [21].

Our contribution. In this paper, we give the first algorithms tackling this problem faster than by the $\mathcal{O}^*(3^n)$ brute-force algorithm obtained by enumerating all legion functions. Both our algorithms rely on structural properties for weak Roman domination functions, described in Section 3. In Section 4, we first give an $\mathcal{O}^*(2^n)$ time and exponential space algorithm. We then show how the exponential space can be avoided by using an exponential algorithm for the Red-Blue Dominating Set problem [26], which leads to an $\mathcal{O}^*(2.2279^n)$ algorithm. In Section 5, we show that the problem can be solved in linear-time on interval graphs.

2. Preliminaries and notations

We consider simple undirected graphs $G = (V, E)$ and assume that $n = |V|$. Given a vertex $v \in V$, we denote by $N(v)$ its *open neighborhood*, by $N[v]$ its *closed neighborhood* (*i.e.* $N[v] = N(v) \cup \{v\}$). For $X \subseteq V$, let $N[X] = \cup_{v\in X} N[v]$ and $N(X) = N[X] \setminus X$. Similarly, given $S \subseteq V$, we use $N_S(v)$ to denote the set $N(v) \cap S$. A subset of vertices $S \subseteq V$ is a *dominating set* of G if for every vertex $v \in V$ either $v \in S$ or $N_S(v) \neq \emptyset$. Furthermore, $Y \subseteq V$ dominates $X \subseteq V$ in $G = (V, E)$ if $X \subseteq N[Y]$. A subset of vertices $S' \subseteq V$ is an *independent set* in G if there is no edge in G between any pair of vertices in S'. Finally, a graph $G = (V, E)$ is *bipartite* whenever its vertex set can be partitioned into two independent sets V_1 and V_2.

2.1. Legion and wrd-functions

A function $f : V \to \{0, 1, 2\}$ is called a *legion function*. With respect to f, a vertex $v \in V$ is said to be *secured* if $f(v) \geqslant 1$, and *unsecured* otherwise. Similarly, a vertex $v \in V$ is said to be *defended* if there exists $u \in N[v]$ such that $f(u) \geqslant 1$. Otherwise, v is said to be *undefended*. The function f is a *weak Roman domination function* (wrd-function for short) if there is no undefended vertex with respect to f, and for every vertex $v \in V$ with $f(v) = 0$ there exists a secured vertex $u \in N(v)$ such that the function $f_{u\to v} : V \to \{0, 1, 2\}$ defined by:

$$f_{u\to v}(x) = \begin{cases} 1 & \text{if } x = v \\ f(u) - 1 & \text{if } x = u \\ f(x) & \text{if } x \notin \{u, v\} \end{cases}$$

has no undefended vertex (see Fig. 1(a)). In other words, $f_{u\to v}$ denotes the legion function obtained by *moving* one legion from u to v.

Given a legion function f, we let V_f^1, V_f^2 denote the sets $\{v \in V : f(v) = 1\}$ and $\{v \in V : f(v) = 2\}$, respectively, and define its *underlying set* as $V_f = V_f^1 \cup V_f^2$. The *cost* of f is then defined by $cost(f) = \sum_{v\in V} f(v) = |V_f^1| + 2|V_f^2|$. Notice that when f is a wrd-function, the set V_f is a (not necessarily minimal) dominating set of G. A pair (V_1, V_2) of subsets of vertices is

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