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Computing square roots of graphs with low maximum degree[☆,☆☆]

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ABSTRACT

A graph H is a square root of a graph G if G can be obtained from H by adding an edge between any two vertices in H that are of distance 2. The Square Root problem is that of deciding whether a given graph admits a square root. This problem is known to be NP-complete for chordal graphs and polynomial-time solvable for non-trivial minor-closed graph classes and a very limited number of other graph classes. We prove that Square Root is $O(n)$-time solvable for graphs of maximum degree 5 and $O(n^4)$-time solvable for graphs of maximum degree at most 6.

1. Introduction

The *square* H^2 of a graph $H = (V_H, E_H)$ is the graph with vertex set V_H, such that any two distinct vertices $u, v \in V_H$ are adjacent in H^2 if and only if u and v are of distance at most 2 in H. In this paper we study the reverse concept: a graph H is a *square root* of a graph G if $G = H^2$. There exist graphs with no square root (such as graphs with a cut-vertex), graphs with a unique square root (such as squares of cycles of length at least 7) as well as graphs with more than one square root (such as complete graphs).

In 1967 Mukhopadhyay [23] characterized the class of connected graphs with a square root. However, in 1994, Motwani and Sudan [22] showed that the decision problem Square Root, which asks whether a given graph admits a square root, is NP-complete. As such, it is natural to restrict the input to special graph classes in order to obtain polynomial-time results. For several well-known graph classes the complexity of Square Root is still unknown. For example, Milanic and Schaudt [21] posed the complexity of Square Root restricted to split graphs and cographs as open problems. In Table 1 we survey the known results.

Rows 6 and 7 in Table 1 correspond to the results in this paper. More specifically, we prove in Section 3 that Square Root is linear-time solvable for graphs of maximum degree at most 5 via a reduction to graphs of bounded treewidth and

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Table 1
The known results for Square Root restricted to some special graph class $\mathcal{G}$. Note that the row for planar graphs is absorbed by the row below. The two unreferenced results are the results of this paper.

Graph class $\mathcal{G}$	Complexity
Planar graphs [19]	Linear
Non-trivial and minor-closed [24]	Linear
K_4-free graphs [12]	Linear
(K_r, P_t)-free graphs [12]	Linear
3-degenerate graphs [12]	Linear
Graphs of maximum degree ≤ 5	Linear
Graphs of maximum degree ≤ 6	Polynomial
Graphs of maximum average degree $< \frac{46}{11}$ [10]	Polynomial
Line graphs [20]	Polynomial
Trivially perfect graphs [21]	Polynomial
Threshold graphs [21]	Polynomial
Chordal graphs [14]	NP-complete

Table 2
The known results for $\mathcal{H}$-Square Root restricted to various graph classes $\mathcal{H}$. The result for 3-sun-free split graphs has been extended to a number of other subclasses of split graphs in [16].

Graph class $\mathcal{H}$	Complexity
Trees [19]	Polynomial
Proper interval graphs [14]	Polynomial
Bipartite graphs [13]	Polynomial
Block graphs [17]	Polynomial
Strongly chordal split graphs [18]	Polynomial
Ptolemaic graphs [15]	Polynomial
3-sun-free split graphs [15]	Polynomial
Cactus graphs [11]	Polynomial
Graphs with girth at least g for any fixed $g \geq 6$ [9]	Polynomial
Graphs of girth at least 5 [8]	NP-complete
Graphs of girth at least 4 [9]	NP-complete
Split graphs [14]	NP-complete
Chordal graphs [14]	NP-complete

in Section 4 that Square Root is $O(n^4)$-time solvable for graphs of maximum degree at most 6 via a reduction to graphs of bounded size.

The $\mathcal{H}$-Square Root problem, which is that of testing whether a given graph has a square root that belongs to some specified graph class $\mathcal{H}$, has also been well studied. We refer to Table 2 for a survey of the known results on $\mathcal{H}$-Square Root.

Finally both Square Root and $\mathcal{H}$-Square Root have been studied under the framework of parameterized complexity. The generalization of Square Root that takes as input a graph G with two subsets R and B of edges that need to be included or excluded, respectively, in any solution (square root)[1] has a kernel of size $O(k)$ for graphs that can be made planar after removing at most k vertices [10]. The problems of testing whether a connected n-vertex graph with m edges has a square root with at most $n-1+k$ edges and whether such a graph has a square root with at least $m-k$ edges are both fixed-parameter tractable when parameterized by k [3].

2. Preliminaries

We only consider finite undirected graphs without loops or multiple edges. We refer to the textbook by Diestel [7] for any undefined graph terminology.

Let G be a graph. We denote the vertex set of G by V_G and the edge set by E_G. The *length* of a path or a cycle is the number of edges of the path or cycle, respectively. The *distance* $\mathrm{dist}_G(u, v)$ between a pair of vertices u and v of G is the number of edges of a shortest path between them. The diameter $\mathrm{diam}(G)$ of G is the maximum distance between two vertices of G. The *neighbourhood* of a vertex $u \in V_G$ is defined as $N_G(u) = \{v \mid uv \in E_G\}$. The *degree* of a vertex $u \in V_G$ is defined as $d_G(u) = |N_G(u)|$. The *maximum degree* of G is $\Delta(G) = \max\{d_G(v) \mid v \in V_G\}$. A vertex of degree 1 and the (unique) edge incident to it are said to be a *pendant* vertex and *pendant* edge of G respectively. A vertex subset of G that consists of mutually adjacent vertices is called a *clique*.

A *tree decomposition* of a graph G is a pair (T, X) where T is a tree and $X = \{X_i \mid i \in V_T\}$ is a collection of subsets (called *bags*) of V_G such that the following three conditions hold:

[1] We give a formal definition of this generalization in Section 4, as we need it for proving that Square Root is $O(n^4)$-time solvable for graphs of maximum degree at most 6.

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