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# Characterizing graphs of maximum matching width at most 2

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ABSTRACT

The maximum matching width is a width-parameter that is defined on a branch-decomposition over the vertex set of a graph. The size of a maximum matching in the bipartite graph is used as a cut-function. In this paper, we characterize the graphs of maximum matching width at most 2 using the minor obstruction set. Also, we compute the exact value of the maximum matching width of a grid.



## 1. Introduction

Treewidth and branchwidth are well-known width-parameters of graphs used in structural graph theory and theoretical computer science. Based on Courcelle's theorem [4], which states that every property on graphs definable in monadic second-order logic can be decided in linear time on a class of graphs with bounded treewidth, many NP-hard problems have been shown to be solvable in polynomial time by dynamic programming when the input has bounded treewidth or branchwidth.

Vatshelle [20] introduced a new graph width-parameter, called the maximum matching width (mm-width in short), that uses the size of a maximum matching as a cut-function in its branch-decomposition of the vertex set of a graph. Maximum matching width is related to treewidth and branchwidth as shown by the inequality $\mathrm{mmw}(G) \leq \max(\mathrm{brw}(G), 1) \leq \mathrm{tw}(G)+1 \leq 3\,\mathrm{mmw}(G)$ for every graph $G$ [20] where $\mathrm{mmw}(G)$, $\mathrm{tw}(G)$, and $\mathrm{brw}(G)$ are the maximum matching width, the treewidth, and the branchwidth of $G$ respectively. This implies that bounding the treewidth or branchwidth is qualitatively equivalent to bounding the maximum matching width. Maximum matching width gives a more efficient algorithm for some problems. For a given branch-decomposition of a graph $G$ of maximum matching width $k$, we can solve the Minimum Dominating Set Problem in time $O^*(8^k)$ [8], which gives a better runtime than $O^*(3^{\mathrm{tw}(G)})$-time algorithm in [19] when $\mathrm{tw}(G) > (\log_3 8)k$. Note that the Minimum Dominating Set Problem cannot be solved in time $O^*((3-\varepsilon)^{\mathrm{tw}(G)})$ for every $\varepsilon > 0$ unless the Strong Exponential Time Hypothesis fails [10].

The Robertson–Seymour theorem [13] states that every minor-closed class of graphs has a finite minor obstruction set. In the other words, a graph $G$ is in the class if and only if $G$ has no minor isomorphic to a graph in the obstruction set. Much work has been done to identify the minor obstruction set for various graph classes, especially for graphs of bounded width-parameters [2,5,9].

Let $K_n$, $C_n$, and $P_n$ be the complete graph, the cycle graph, and the path graph on $n$ vertices, respectively. The graphs $K_3$ and $K_4$ are the unique minor obstruction for the graphs of treewidth at most 1 and 2 [21], respectively. The minor obstruction set for the class of graphs having treewidth at most 3 is $\{K_5, K_{2,2,2}, K_2 \times C_5, M_8\}$ where $K_2 \times C_5$ is the Cartesian product of $K_2$ and $C_5$, and $M_8$ is the Wagner graph, also called the Möbius ladder with eight vertices [1,16].

Robertson and Seymour [12] gave a characterization for the classes of graphs of branchwidth at most 1 and at most 2. The graphs $K_3$ and $P_4$ are forbidden minors for the graphs of branchwidth at most 1. For the class of graphs of branchwidth

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at most 2, its minor obstruction is the same as treewidth, which is $K_4$. The graphs of branchwidth at most 3 have four minor obstructions; $\{K_5, K_{2,2,2}, K_2 \times C_4, M_8\}$ [3].

One of the main results of this paper is to find the minor obstruction set for the class of graphs of mm-width at most 2. Note that the class of graphs with bounded mm-width is closed under taking minor, as shown in Corollary 2.3. Our main result is the following.

**Theorem 3.17.** *Let $\mathcal{O} = \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$ be the set of 42 graphs in Figs. 1, 5–7. A graph G has mm-width at most 2 if and only if G has no minor isomorphic to a graph in $\mathcal{O}$.*

The exact value of some width-parameters for grid graphs is well known. For an integer $k \geq 1$, the branchwidth and treewidth of the $k \times k$-grid are $k$ [12,17], and the rank-width of the $k \times k$-grid is $k - 1$ [7]. From the inequality $\text{rw}(G) \leq \text{mmw}(G) \leq \max(\text{brw}(G), 1)$ [20], the mm-width of the $k \times k$-grid is either $k - 1$ or $k$. Our second result is that the latter is the right answer when $k \geq 2$.

**Theorem 4.7.** *The $k \times k$-grid has mm-width $k$ for $k \geq 2$.*

Section 2 lists some of the definitions, including a *tangle*, and provides preliminaries for the maximum matching width. In Section 3 we identify the minor obstruction set for graphs with mm-width at most 2. Section 4 gives the result for the precise mm-width of the square grids.

## 2. Preliminaries

Every graph $G = (V, E)$ in this paper is finite and simple. For a set $X \subseteq V(G) \cup E(G)$, we write $G \setminus X$ to denote the graph obtained from $G$ by deleting all vertices and edges in $X$. If $X \subseteq E(G)$, we write $G/X$ to denote the graph obtained from $G$ by contracting the edges in $X$. If $X = \{x\}$, then we write $G \setminus x$ and $G/x$ instead of $G \setminus X$ and $G/X$, respectively. If a subgraph $G'$ of $G$ with $V(G') = X$ contains all the edges of $G$ whose both ends are in $X$, then we call $G'$ *induced by* $X$ and write $G' := G[X]$. For a graph $G$ and disjoint subsets $X, Y \subseteq V(G)$, let $E_G[X, Y]$ (or $E[X, Y]$) denote the set of all edges $e = uv$ where $u$ is in $X$ and $v$ is in $Y$, and let $G[X, Y] = G(X \cup Y, E[X, Y])$. A graph $G$ is *k-connected* if $|V(G)| \geq k$ and $G \setminus X$ is connected for every $X \subset V(G)$ with $|X| < k$. A *bridge* is an edge $e$ such that $G \setminus e$ has more components than $G$. A *block* is either a bridge as a subgraph or a maximal 2-connected subgraph.

We say that a tree is *ternary* if all vertices have degree 1 or 3. A *branch-decomposition* of a finite set $X$ is a pair $(T, \mathcal{L})$ of a ternary tree $T$ together with a bijection $\mathcal{L}$ from the leaves of $T$ to $X$. Note that an edge $ab$ of $T$ partitions the leaves of $T$ into two parts, say $A$ and $B$. We say an edge $e$ *induces* the partition $(A, B)$. A function $f : 2^X \to \mathbb{Z}$ is *symmetric* if $f(A) = f(X \setminus A)$ for all $A \subseteq X$, and the function $f$ is *submodular* if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq X$. For each edge $e$ of $T$, and a symmetric, submodular function $f$, the *f-value* of $e$ is equal to $f(A) = f(B)$ where $(A, B)$ is the partition induced by $e$. The *f-width* of a branch-decomposition $(T, \mathcal{L})$ is the maximum $f$-value of an edge of $T$, and the *f-width* of $X$ is the minimum value of the $f$-width over all possible branch-decompositions of $X$. This notion of $f$-width provides a link between several width parameters.

For $A \subseteq E(G)$, let $br : 2^{E(G)} \to \mathbb{Z}$ be the function so that $br(A)$ is the number of vertices that are incident to both an edge in $A$ and an edge in $E(G) \setminus A$. The *branchwidth* of $G$, denoted by $\text{brw}(G)$, is the *br-width of* $E(G)$.

For $A \subseteq V(G)$, let $r : 2^{V(G)} \to \mathbb{Z}$ be the function such that $r(A)$ is the rank of the adjacency matrix between $A$ and $V(G) \setminus A$ over $\mathbb{F}_2$. The *rank-width* of $G$, denoted by $\text{rw}(G)$, is the *r-width of* $V(G)$.

Let $\text{mm}_G : 2^{V(G)} \to \mathbb{Z}$ be the function such that $\text{mm}_G(A)$ is the size of a maximum matching in $G[A, V(G) \setminus A]$. Note that the function $\text{mm}_G$ is symmetric and submodular [15]. We use mm instead of $\text{mm}_G$ if the host graph $G$ is clear from the context. The *maximum matching width* of $G$, denoted by $\text{mmw}(G)$, is the mm-width of $V(G)$.

A graph $H$ is a *minor* of a graph $G$ if $H$ can be constructed from $G$ by deleting edges, deleting vertices, and contracting edges. We call a graph $G$ *minor-minimal* with respect to a property $\mathcal{P}$ if $G$ has $\mathcal{P}$ but no proper minor of $G$ has $\mathcal{P}$. A graph $G$ is a *forbidden minor* of a graph class $\mathcal{C}$ if $H \notin \mathcal{C}$ whenever $H$ has a minor isomorphic to $G$. Robertson and Seymour [13] state that the collection of minor-minimal graphs outside a minor-closed graph class is finite. The collection is called the *minor obstruction set*.

A graph is *chordal* if every induced cycle in the graph has length 3. A *chordalization* of a graph $G$ is a chordal graph $H$ such that $V(H) = V(G)$ and $E(G) \subseteq E(H)$. An *intersection graph* $G$ over a family $\{A_i\}$ of sets is the graph with $V(G) = \{A_i\}$ and $E(G) = \{A_iA_j : A_i \cap A_j \neq \emptyset\}$. Remark that a graph is chordal if and only if it is the intersection graph of the edge sets of subtrees of a tree [6].

### 2.1. Maximum matching width

Jeong, Sæther, and Telle [8] gave a new characterization of graphs of mm-width at most $k$ as an intersection graph by the following theorem. A tree is called *nontrivial* if it has at least one edge and a tree is *subcubic* if all vertices have degree at most 3.

**Theorem 2.1** ([8]). *The maximum matching width of a graph G is at most k if and only if there exist a subcubic tree T and a set $\{T_x\}_{x \in V(G)}$ of nontrivial subtrees of T such that*