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Maximum matching width: New characterizations and a fast algorithm for dominating set

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ABSTRACT

A graph of treewidth k has a representation by subtrees of a ternary tree, with subtrees of adjacent vertices sharing a tree node, and any tree node sharing at most $k + 1$ subtrees. Likewise for branchwidth, but with a shift to the edges of the tree rather than the nodes. In this paper we show that the mm-width of a graph – maximum matching width – combines aspects of both these representations, targeting tree nodes for adjacency and tree edges for the parameter value. The proof of this new characterization of mm-width is based on a definition of canonical minimum vertex covers of bipartite graphs. We show that these behave in a monotone way along branch decompositions over the vertex set of a graph.

We use these representations to compare mm-width with treewidth and branchwidth, and also to give another new characterization of mm-width, by subgraphs of chordal graphs. We prove that given a graph G and a branch decomposition of maximum matching width k we can solve the Minimum Dominating Set Problem in time $O^*(8^k)$, thereby beating $O^*(3^{\mathrm{tw}(G)})$ whenever $\mathrm{tw}(G) > \log_3 8 \times k \approx 1.893k$. Note that $\mathrm{mmw}(G) \leq \mathrm{tw}(G) + 1 \leq 3\,\mathrm{mmw}(G)$ and these inequalities are tight. Given only the graph G and using the best known algorithms to find decompositions, maximum matching width will be better for Minimum Dominating Set whenever $\mathrm{tw}(G) > 1.549 \times \mathrm{mmw}(G)$.

1. Introduction

The treewidth $\mathrm{tw}(G)$ and branchwidth $\mathrm{bw}(G)$ of a graph G are connectivity parameters of importance in algorithm design. By dynamic programming along the associated tree decomposition or branch decomposition one can solve many graph optimization problems in time linear in the graph size and exponential in the parameter. For every graph G, its treewidth and branchwidth are related by $\mathrm{bw}(G) \leq \mathrm{tw}(G) + 1 \leq \frac{3}{2}\,\mathrm{bw}(G)$ [18]. The two parameters are thus equivalent with respect to fixed parameter tractability (FPT), with a problem being FPT parameterized by treewidth if and only if it is FPT parameterized by branchwidth. For some of these problems the best known FPT algorithms are optimal, up to some complexity theoretic assumption. For example, Minimum Dominating Set Problem can be solved in time $O^*(3^{\mathrm{tw}(G)})$ when given a tree decomposition of width $\mathrm{tw}(G)$ [21] but not in time $O^*((3-\varepsilon)^{\mathrm{tw}(G)})$ for every $\varepsilon > 0$ unless the Strong Exponential Time Hypothesis (SETH) fails [13].

Recently, a graph parameter equivalent to treewidth and branchwidth was introduced, the maximum matching width (or mm-width) $\mathrm{mmw}(G)$, defined by a branch decomposition over the vertex set of a graph G, using the symmetric submodular cut function obtained by taking the size of a maximum matching of the bipartite graph crossing the cut (by König's Theorem

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equivalent to minimum vertex cover) [22]. For a graph G we have $\text{mmw}(G) \leq \text{bw}(G) \leq \text{tw}(G) + 1 \leq 3\,\text{mmw}(G)$ and these inequalities are tight [22], for example any balanced branch decomposition will show that $\text{mmw}(K_n) = \lceil \frac{n}{3} \rceil$ and it is known that $\text{tw}(K_n) = n - 1$ and $\text{bw}(G_{n\times n}) = \text{mmw}(G_{n\times n}) = n$ where $G_{n\times n}$ is the $n \times n$-grid graph [18,10].

In this paper we show that given a branch decomposition over the vertex set of mm-width k we can solve Minimum Dominating Set Problem in time $O^*(8^k)$. This runtime beats the $O^*(3^{\text{tw}(G)})$-time algorithm for treewidth [21] whenever $\text{tw}(G) > \log_3 8 \times k \approx 1.893k$. If we assume only a graph G as input, then since mm-width has a symmetric submodular cut function [20] we can approximate mm-width to within a factor $3\,\text{mmw}(G) + 1$ in $O^*(2^{3\,\text{mmw}(G)})$-time using the generic algorithm of [15], giving a total runtime for solving Minimum Dominating Set Problem of $O^*(2^{9\,\text{mmw}(G)})$. For treewidth we can in $O^*(2^{3.7\,\text{tw}(G)})$-time [1] get an approximation to within a factor $(3 + 2/3)\,\text{tw}(G)$ giving a total runtime for solving Minimum Dominating Set Problem of $O^*(3^{3.666\,\text{tw}(G)})$.[1] This implies that on input G, using maximum matching width gives better exponential factor whenever $\text{tw}(G) > 1.549\,\text{mmw}(G)$.

Our results are based on a new characterization of graphs of mm-width at most k, as intersection graphs of subtrees of a tree. It can be formulated as follows:

For each $k \geq 2$ a graph G on vertices $v_1, v_2, \ldots, v_n$ has $\underline{\text{mmw}(G) \leq k}$ if and only if there exists a tree T of maximum degree at most 3 with nontrivial subtrees $T_1, T_2, \ldots, T_n$ such that if $v_iv_j \in E(G)$ then subtrees T_i and T_j have at least one $\underline{\text{node}}$ of T in common and for each $\underline{\text{edge}}$ of T there are at most k subtrees using it.

Replacing the three underlined parts in the above characterization by ($\underline{\text{tw}(G) \leq k-1}$, $\underline{\text{node}}$, $\underline{\text{node}}$) we define treewidth, while replacing by ($\underline{\text{bw}(G) \leq k}$, $\underline{\text{edge}}$, $\underline{\text{edge}}$) we define branchwidth [8,19,2,16]. Note that while treewidth has a focus on nodes and branchwidth a focus on edges, mm-width combines aspects of both by a partial focus on nodes and on edges.

In this way the maximum matching width can more easily be compared to the much studied graph parameters treewidth and branchwidth. In our Theorem 3.10 we do this when we show $\text{bw}(G) \leq 2\,\text{mmw}(G)$, improving on the previous bound of $\text{bw}(G) \leq 3\,\text{mmw}(G)$ from [22]. Since the proof of $\text{tw}(G) \leq 3\,\text{mmw}(G) - 1$ in [22] was based on a non-monotone cops and robber strategy not known to be efficiently computable, Vatshelle [22] asked whether one can find, in time $O(n^{3.5})$, a tree decomposition of width at most $3k - 1$ given a branch decomposition of mm-width k. Using the new characterization, we resolve this question in Theorem 3.9. We also arrive at the following alternative characterization: a graph G has $\text{mmw}(G) \leq k$ if and only if it is a subgraph of a chordal graph H and for every maximal clique X of H there exist $A, B, C \subseteq X$ with $A \cup B \cup C = X$ and $|A|, |B|, |C| \leq k$ such that each subset of X that is a minimal separator of H is a subset of either A, B or C.

In Section 2 we give definitions. In Section 3 we define canonical minimum vertex covers for all bipartite graphs, show some monotonicity properties of these, and use these properties to give the new characterizations of mm-width. In Section 4 we give the dynamic programming algorithm for Minimum Dominating Set Problem. We end in Section 5 with some discussions.

2. Definitions

For a simple and loopless graph $G = (V, E)$ and its vertex v, let $N(v)$ be the set of all vertices adjacent to v in G, and $N[v] = N(v) \cup \{v\}$. For a subset S of $V(G)$, let $N(S)$ be the set of all vertices that are not in S but are adjacent to some vertex of S in G, and $N[S] = N(S) \cup S$. A subset of vertices $S \subseteq V(G)$ is said to *dominate* the vertices in $N[S]$, and it is a *dominating set* of G if $N[S] = V(G)$. For disjoint $A, B \subseteq V$ we denote by $G[A, B]$ the bipartite subgraph of G containing all edges between a vertex in A and a vertex in B.

A *tree decomposition* of a graph G is a pair $(T, \{X_t\}_{t\in V(T)})$ consisting of a tree T and a family $\{X_t\}_{t\in V(T)}$ of vertex sets $X_t \subseteq V(G)$, called *bags*, satisfying the following three conditions:

(1) each vertex of G is in at least one bag,
(2) for each edge uv of G, there exists a bag that contains both u and v, and
(3) for nodes u, v, w of T, if v is on the path from u to w, then $X_u \cap X_w \subseteq X_v$.

The *width* of a tree decomposition $(T, \{X_t\}_{t\in V(T)})$ is $\max_{t\in V(T)} |X_t| - 1$. The *treewidth* of G, denoted by $\text{tw}(G)$, is the minimum width over all possible tree decompositions of G.

A *branch decomposition* over a finite set X, for some set of elements X, is a pair (T, δ) where T is a tree of maximum degree at most 3, and δ is a bijection from the leaves of T to the elements in X. Each edge ab disconnects T into two subtrees T_a and T_b. Likewise, each edge ab of T partitions the elements of X into two parts A and B, namely the elements mapped by δ from the leaves in T_a, and in T_b, respectively. An edge $ab \in E(T)$ is said to *induce* the partition $\{A, B\}$ of X.

A *rooted branch decomposition* over a finite set X is a branch decomposition (T, δ) over X where we subdivide an edge of T and make the new node the *root* r. In a rooted branch decomposition, for an internal node $x \in V(T)$, we denote by $\delta(x)$ the union of $\delta(\ell)$ for all leaves ℓ having x as ancestor. Let $\overline{\delta(x)} = X \setminus \delta(x)$ be the complement of $\delta(x)$.

For example, $\delta(x) = \{\delta(\ell_1), \delta(\ell_2), \delta(\ell_3)\}$ in Fig. 1.

For a finite set X and for all $A, B \subseteq X$, a function $f : 2^X \to \mathbb{R}$ is *symmetric* if $f(A) = f(X \setminus A)$ and *submodular* if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. Given a symmetric submodular function $f : 2^X \to \mathbb{R}$, using branch decompositions

[1] Note that there is also an $O^*(c^{\text{tw}(G)})$ time 3-approximation of treewidth [4], but the c is so large that the approximation alone has a bigger exponential part than the entire Minimum Dominating Set algorithm when using the 3.666-approximation.

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