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Infinitely many minimal classes of graphs of unbounded clique-width ☆

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ABSTRACT

The celebrated theorem of Robertson and Seymour states that in the family of minor-closed graph classes, there is a unique minimal class of graphs of unbounded tree-width, namely, the class of planar graphs. In the case of tree-width, the restriction to minor-closed classes is justified by the fact that the tree-width of a graph is never smaller than the tree-width of any of its minors. This, however, is not the case with respect to clique-width, as the clique-width of a graph can be (much) smaller than the clique-width of its minor. On the other hand, the clique-width of a graph is never smaller than the clique-width of any of its induced subgraphs, which allows us to be restricted to hereditary classes (that is, classes closed under taking induced subgraphs), when we study clique-width. Up to date, only finitely many minimal hereditary classes of graphs of unbounded clique-width have been discovered in the literature. In the present paper, we prove that the family of such classes is infinite. Moreover, we show that the same is true with respect to linear clique-width.

1. Introduction

Clique-width is a graph parameter which is important in theoretical computer science, because many algorithmic problems that are generally NP-hard become polynomial-time solvable when restricted to graphs of bounded clique-width [4]. Clique-width is a relatively new notion and it generalises another important graph parameter, tree-width, studied in the literature for decades. Clique-width is stronger than tree-width in the sense that graphs of bounded tree-width have bounded clique-width, but not necessarily vice versa. For instance, both parameters are bounded for trees, while for complete graphs only clique-width is bounded.

When we study classes of graphs of bounded tree-width, we may assume without loss of generality that together with every graph G our class contains all minors of G, as the tree-width of a minor can never be larger than the tree-width of the graph itself. In other words, when we try to identify classes of graphs of bounded tree-width, we may restrict ourselves to minor-closed graph classes. However, when we deal with clique-width this restriction is not justified, as the clique-width of a minor of G can be much larger than the clique-width of G. On the other hand, the clique-width of G is never smaller than the clique-width of any of its induced subgraphs [5]. This allows us to be restricted to hereditary classes, that is, those that are closed under taking induced subgraphs.

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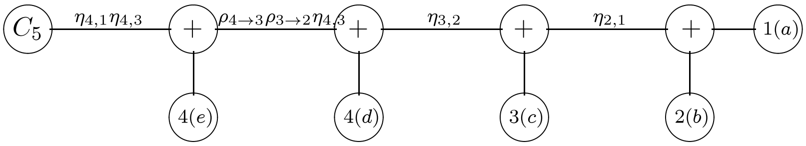

Fig. 1. The tree representing the expression defining a C_5.

One of the most remarkable outcomes of the graph minor project of Robertson and Seymour is the proof of Wagner's conjecture stating that the minor relation is a well-quasi-order [13]. This implies, in particular, that in the world of minor-closed graph classes there exist minimal classes of unbounded tree-width and the number of such classes is finite. In fact, there is just one such class (the planar graphs), which was shown even before the proof of Wagner's conjecture [12].

In the world of hereditary classes the situation is more complicated, because the induced subgraph relation is not a well-quasi-order. It contains infinite antichains, and hence, there may exist infinite strictly decreasing sequences of graph classes with no minimal one. In other words, even the existence of minimal hereditary classes of unbounded clique-width is not an obvious fact. This fact was recently confirmed in [8]. However, whether the number of such classes is finite or infinite remained an open question. In the present paper, we settle this question by showing that the family of minimal hereditary classes of unbounded clique-width is infinite. Moreover, we prove that the same is true with respect to *linear* clique-width.

The organisation of the paper is as follows. In the next section, we introduce basic notation and terminology. In Section 3, we describe a family of graph classes of unbounded clique-width and prove that infinitely many of them are minimal with respect to this property. In Section 4, we identify more classes of unbounded clique-width. Finally, Section 5 concludes the paper with a number of open problems.

2. Preliminaries

All graphs in this paper are undirected, without loops and multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. The *neighbourhood* of a vertex $v \in V(G)$ is the set of vertices adjacent to v and the *degree* of v is the size of its neighbourhood. As usual, by P_n and C_n we denote a chordless path and a chordless cycle with n vertices, respectively.

In a graph, an *independent set* is a subset of vertices no two of which are adjacent. A graph is *bipartite* if its vertices can be partitioned into two independent sets. Given a bipartite graph G together with a bipartition of its vertices into two independent sets V_1 and V_2, the *bipartite complement* of G is the bipartite graph obtained from G by complementing the edges *between* V_1 and V_2.

Let G be a graph and $U \subseteq V(G)$ a subset of its vertices. Two vertices of U will be called U-*similar* if they have the same neighbourhood outside U. Clearly, U-similarity is an equivalence relation. The number of equivalence classes of U-similarity will be denoted $\mu(U)$. Also, by $G[U]$ we will denote the subgraph of G induced by U, that is, the subgraph of G with vertex set U and two vertices being adjacent in $G[U]$ if and only if they are adjacent in G. We say that a graph H is an *induced subgraph* of G if H is isomorphic to $G[U]$ for some $U \subseteq V(G)$.

A class X of graphs is *hereditary* if it is closed under taking induced subgraphs, that is, $G \in X$ implies $H \in X$ for every induced subgraph H of G. It is well-known that a class of graphs is hereditary if and only if it can be characterised in terms of forbidden induced subgraphs. More formally, given a set of graphs M, we say that a graph G is M-*free* if G does not contain induced subgraphs isomorphic to graphs in M. Then a class X is hereditary if and only if graphs in X are M-free for a set M.

The notion of clique-width of a graph was introduced in [3]. The clique-width of a graph G is denoted $\mathrm{cwd}(G)$ and is defined as the minimum number of labels needed to construct G by means of the following four graph operations:

- creation of a new vertex v with label i (denoted $i(v)$),
- disjoint union of two labelled graphs G and H (denoted $G \oplus H$),
- connecting vertices with specified labels i and j (denoted $\eta_{i,j}$) and
- renaming label i to label j (denoted $\rho_{i\to j}$).

Every graph can be defined by an algebraic expression using the four operations above. This expression is called a k-expression if it uses k different labels. For instance, the cycle C_5 on vertices a, b, c, d, e (listed along the cycle) can be defined by the following 4-expression:

$$\eta_{4,1}(\eta_{4,3}(4(e) \oplus \rho_{4\to3}(\rho_{3\to2}(\eta_{4,3}(4(d) \oplus \eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))))).$$

Alternatively, any algebraic expression defining G can be represented as a rooted tree, whose leaves correspond to the operations of vertex creation, the internal nodes correspond to the $\oplus$-operations, and the root is associated with G. The operations η and ρ are assigned to the respective edges of the tree. Fig. 1 shows the tree representing the above expression defining a C_5.

Let us observe that the tree in Fig. 1 has a special form known as a *caterpillar tree* (that is, a tree that becomes a path after the removal of vertices of degree 1). The minimum number of labels needed to construct a graph G by means of caterpillar

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