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An energy stable method for the Swift–Hohenberg equation with quadratic–cubic nonlinearity

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Highlights

- We present temporally first- and second-order accurate methods for the SH equation with quadratic–cubic nonlinearity.
- The unconditional unique solvability and unconditional stability of the proposed methods are analytically proven.
- We numerically observe that pattern formation in the SH equation is mainly affected by its quadratic term.

Abstract

We present temporally first- and second-order accurate methods for the Swift–Hohenberg (SH) equation with quadratic–cubic nonlinearity. In order to handle the nonconvex, nonconcave term in the energy for the SH equation, we add an auxiliary term to make the combined term convex, which yields a convex–concave decomposition of the energy. As a result, the first- and second-order methods are unconditionally uniquely solvable and unconditionally stable with respect to the energy and pseudoenergy of the SH equation, respectively. And the Fourier spectral method is used for the spatial discretization. We present numerical examples showing the accuracy and energy stability of the proposed methods and the effect of the quadratic term in the SH equation on pattern formation.

Keywords: Swift–Hohenberg equation; Unique solvability; Energy stability; Fourier spectral method; Pattern formation

1. Introduction

In this paper, we consider the following Swift–Hohenberg (SH) equation [1,2]:

$$\frac{\partial \phi}{\partial t} = -\left(\phi^3 - g\phi^2 + \left(-\epsilon + (1+\Delta)^2\right)\phi\right), \tag{1}$$

where ϕ is the density field and $g \geq 0$ and $\epsilon > 0$ are constants with physical significance. We assume that ϕ and $\Delta\phi$ are periodic on a domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$). The SH equation has been widely used as a model for the study of

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pattern formation [3–8] and is an L^2-gradient flow for the following free energy functional [1,2]:

$$\mathcal{E}(\phi) := \int_\Omega \left(\frac{1}{4}\phi^4 - \frac{g}{3}\phi^3 + \frac{1}{2}\phi\left(-\epsilon + (1+\Delta)^2\right)\phi \right) d\mathbf{x}, \tag{2}$$

i.e., $\frac{\partial \phi}{\partial t} = -\frac{\delta \mathcal{E}}{\delta \phi}$, where $\frac{\delta}{\delta \phi}$ denotes the variational derivative. Thus, the energy (2) is nonincreasing in time:

$$\frac{d\mathcal{E}}{dt} = \int_\Omega \frac{\delta \mathcal{E}}{\delta \phi}\frac{\partial \phi}{\partial t}\, d\mathbf{x} = -\int_\Omega \left(\frac{\partial \phi}{\partial t}\right)^2 d\mathbf{x} \le 0.$$

The SH equation is a fourth-order nonlinear partial differential equation and cannot generally be solved analytically. Therefore, accurate and efficient numerical methods are desirable. One criterion for developing a numerical method for the SH equation is whether the method inherits the energy dissipation property of the SH equation. There are various related works [9–15] but most of them considered the case of $g = 0$ except [11,13].

In this paper, we propose temporally first- and second-order accurate methods for the SH equation with both $g = 0$ and $g > 0$. We use the convex splitting idea [16,17], in which convex and concave parts of the energy are treated implicitly and explicitly, respectively. For the SH equation with $g = 0$, (2) has a straightforward convex–concave splitting [10]. However, there is a difficulty with $g > 0$ since $-\frac{g}{3}\int_\Omega \phi^3\, d\mathbf{x}$ in (2) is neither convex nor concave. In order to apply the convex splitting idea for both $g = 0$ and $g > 0$, we add and subtract an auxiliary term in (2). Then, a convex–concave decomposition is available. The first-order method is automatically obtained applying the convex splitting idea, and we use a second-order secant type approach to the convex part and a second-order extrapolation to the concave part for the second-order method. We show analytically that the first- and second-order methods are unconditionally uniquely solvable and unconditionally stable with respect to the energy and pseudoenergy of the SH equation, respectively. And the Fourier spectral method [14,18–20] is used for the spatial discretization.

This paper is organized as follows. In Section 2, we describe the convex splitting with an auxiliary term. In Sections 3 and 4, we propose first- and second-order methods for the SH equation, respectively, with proofs of unconditional unique solvability and unconditional energy stability and description of numerical implementation. In Section 5, we present numerical examples showing the accuracy and energy stability of the proposed methods and the effect of g on pattern formation. Finally, conclusions are given in Section 6.

2. Convex–concave energy decomposition with an auxiliary term

We propose to split the energy (2) according to $\mathcal{E}(\phi) = \mathcal{E}_c(\phi) - \mathcal{E}_e(\phi)$ with

$$\mathcal{E}_c(\phi) = \int_\Omega \left(\frac{1}{4}\phi^4 - \frac{g}{3}\phi^3 + \frac{A}{2}\phi^2 + \frac{1}{2}\phi(1+\Delta)^2\phi \right) d\mathbf{x}, \quad \mathcal{E}_e(\phi) = \int_\Omega \frac{A+\epsilon}{2}\phi^2\, d\mathbf{x}. \tag{3}$$

Lemma 1. *Suppose that ϕ and $\Delta\phi$ are sufficiently regular. Then, both $\mathcal{E}_c(\phi)$ and $\mathcal{E}_e(\phi)$ in (3) are convex provided $A \ge \frac{g^2}{3}$.*

Proof. Suppose that ψ and $\Delta\psi$ are sufficiently regular. For $\mathcal{E}_c(\phi)$,

$$\begin{aligned}\mathcal{E}_c(\phi + s\psi) &= \mathcal{E}_c(\phi) + s\frac{\delta \mathcal{E}_c(\phi)}{\delta \phi} + \frac{s^2}{2}\frac{\delta^2 \mathcal{E}_c(\phi)}{\delta \phi^2} + O(s^3) \\ &= \mathcal{E}_c(\phi) + s\int_\Omega \left(\phi^3 - g\phi^2 + A\phi + (1+\Delta)^2\phi\right)\psi\, d\mathbf{x} \\ &\quad + \frac{s^2}{2}\int_\Omega \left((3\phi^2 - 2g\phi + A)\psi^2 + \psi(1+\Delta)^2\psi\right) d\mathbf{x} + O(s^3).\end{aligned}$$

Then, we obtain

$$\begin{aligned}\left.\frac{d^2\mathcal{E}_c(\phi + s\psi)}{ds^2}\right|_{s=0} &= \int_\Omega \left(3\left(\phi - \frac{g}{3}\right)^2\psi^2 + \left(A - \frac{g^2}{3}\right)\psi^2 + ((1+\Delta)\psi)^2\right) d\mathbf{x} \\ &\ge 0 \qquad \text{if } A \ge \frac{g^2}{3}.\end{aligned}$$