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Complexity modeling and analysis of chaos and other fluctuating phenomena

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ABSTRACT

The refined composite multiscale-entropy algorithm was applied to the time-dependent behavior of the Weierstrass functions, colored noise, and Logistic map to provide the fresh insight into the dynamics of these fluctuating phenomena. For the Weierstrass function, the complexity of fluctuations was found to increase with respect to the fractional dimension, *D*, of the graph. Additionally, the sample-entropy curves increased in an exponential fashion with increasing *D*. This increase in the complexity was found to correspond to a rising amount of irregularities in the oscillations. In terms of the colored noise, the complexity of the fluctuations was found to be the highest for the $1/f$ noise (f is the frequency of the generated noise), which is in agreement with findings in the literature. Moreover, the sample-entropy curves exhibited a decreasing trend for noise when the spectral exponent, β, was less than 1 and obeyed an increasing trend when $\beta > 1$. Importantly, a direct relationship was observed between the power-law exponents for the curves and the spectral exponents of the noise. For the logistic map, a correspondence was observed between the complexity maps and its bifurcation diagrams. Specifically, the map of the sample-entropy curves was negligible, when the bifurcation parameter, *R*, varied between 3 and 3.5. Beyond these values, the curves attained non-zero values that increased with increasing *R*, in general.

1. Introduction

A variety of the sample-entropy (Sample En.) techniques have been proposed to study the complexity of time-series data representing nonlinear dynamical systems [1]. One such technique is the ApEn algorithm [2–4], which measures the probability that similar sequences (for a given number of points) will remain like each other when an additional point is added. However, this method contains bias due to self-matching. To overcome this issue, the SampEn technique [5,6], which excludes self-matching in the calculation, was proposed by Richman et al. [7]. Here the SampEn is defined as the negative natural logarithm of the conditional probability that two sequences remain similar at the next point.

The multiscale entropy (MSE) algorithm was proposed by Costa et al. [8] to calculate SampEn over a range of scales to represent the complexity of a time series. Importantly, the MSE algorithm resolved an issue with the ApEn method, which stated that the white noise consisted of fluctuations that were more complex than those associated with the $1/f$ noise [9]. Here, f is defined as the frequency of the generated noise, which is bounded between arbitrarily small and large values. However, this result was contradictory since the $1/f$ noise was thought to be more intricate in nature. However, the MSE technique, as proposed by Costa et al., showed that although the white noise was more complex at lower scales, the $1/f$ noise possessed higher levels of complexity at larger scaling factors [8,10].

In addition, the MSE algorithm has been found to be useful in analyzing and modeling temporal data, such as the serrated flow [11,12], during mechanical deformation, in different alloy systems [13–15], physiological-time series [8,10,16–19], bearing vibration data [20], mechanical fault diagnosis [21], and financial time series [22,23]. However, the MSE technique does have issues, such as problems in accuracy and validity at large scale factors [9]. To tackle these issues, Wu et al. [24] developed the composite multiscale entropy (CMSE) algorithm, which can estimate the complexity more accurately but increases the chance of producing undefined values. This technique has since been used to analyze financial-time series [25,26].

More recently, Wu et al. modified the CMSE algorithm slightly to produce what is known as the refined composite multiscale en-

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tropy (RCMSE) algorithm [9]. In their work, they compared the complexity of the white and 1/f noise. In terms of accuracy, it was found that the RCMSE algorithm outperformed both the MSE and CMSE algorithms. Like its predecessors, this technique has been used to study the complexity of different phenomena such as physiological systems [27,28] and the intrinsic dynamics of traffic signals [29].

Therefore, the goal of the present work is to use the RCMSE method to model and analyze the complexity of different fluctuating phenomena. These phenomena include the colored noise, the Weierstrass function, and the logistic map. In terms of the colored noise, the current study will expand upon the studies conducted by [8,10,15,30] on the white and $1/f$ noise, where the noise with spectral exponents ranging from –2 to 2 will be modeled and analyzed. Furthermore, this study will provide an innovative way to understand how the regularity of a fractal function changes with respect to its fractional dimension. This investigation also takes an original approach to examining the logistic map, where the complexity of its fluctuations will be examined with respect to its chaotic behavior. Therefore, the present work is significant since it advances our fundamental understanding of the above phenomena.

2. Refined composite multiscale entropy modeling and analysis

For this section, the methodology of [9] will be used. Given a discrete time series of the form, $X = [\ x_1\ x_2\ \dots\ x_i\ \dots\ x_N\ ]$, one constructs the coarse-grained (averaged) time series, $y^{\tau}_{j,k}$, using Eq. (1), which is written as:

$$y^{\tau}_{k,j} = \frac{1}{\tau}\sum_{i=(j-1)\tau+k}^{j\tau+k-1} x_i \quad ; \quad 1 \le j \le \frac{N}{\tau} \quad 1 \le k \le \tau \tag{1}$$

Here N is the total number of points in the original data set, and k is an indexing factor, which dictates at which x_i one begins the coarse-graining procedure. Additionally, one should note that the coarse-grained series, $y^1_{1,1}$, is simply the original time series, X. Fig. 1 gives a schematic illustration of the coarse-graining procedure. At this point, one constructs the template vectors, $y^{\tau,m}_i$, of dimension, m [8]:

$$y^{\tau,m}_i = \left\{ y^{\tau}_i\ y^{\tau}_{i+1} \dots . y^{\tau}_j \dots . y^{\tau}_{i+m-1} \right\} \quad ; \quad 1 \le i \le N - m \tag{2}$$

Once $y^{\tau}_{k,j}$ is constructed, the next step is to write the time series of y^{τ}_k as a vector for each scale factor, τ:

$$y^{\tau}_k = \left\{ y^{\tau}_{k,1}\, y^{\tau}_{k,2} \dots y^{\tau}_{k,N} \right\} \tag{3}$$

The next step in the process is to find n matching sets of distinct template vectors. It should be noted that the previous studies used $m = 2$ as the size of the template vector [7–9]. For two vectors to match, the infinity norm, $d^{\tau,m}_{jk}$, of the difference between them must be less than a predefined tolerance value, r. Here the infinity norm may be written as:

$$\begin{aligned} d^{\tau,m}_{jk} &= \left\| \mathbf{y}^{\tau,m}_j - \mathbf{y}^{\tau,m}_k \right\|_{\infty} \\ &= \max\left\{ \left| y^{\tau}_{1,j} - y^{\tau}_{1,k} \right| \dots \left| y^{\tau}_{i+m-1,j} - y^{\tau}_{i+m-1,k} \right| \right\} < r \end{aligned} \tag{4}$$

Typically, r is chosen as 0.1–0.2 times the standard deviation, of the original data set [10]. This choice ensures that the sample entropy relies on the sequential ordering, and not the variance, of the original time series. For this study, a value of $r = 0.15\sigma$ will be used.

Fig. 2 illustrates the matching process for the coarse-grained series, $y^1_{1,j} = X(j)$ (here $k = 1$) [10]. In the graph, there is the template sequence, {x(1), x(2), x(3)}, which matches the template sequence, {x(28), x(29), x(30)}, meaning that there is a matching three-component template set. Here the matching points for the three-component templates are denoted by blue boxes in the figure. This calculation is, then, repeated for the next three-component template sequence in which a total count of matching template sequences is taken. Then the entire process is repeated for all two-component template sequences. The number of matching two- and three component template sequences are again summed and added to the cumulative total.

This procedure is performed for each k from 1 to τ and, then, the number of matching template sequences, n^m_k and n^{m+1}_k, is summed, which is written as:

$$\mathrm{RCMSE}(\mathbf{y}, \tau, m, r) = Ln\left(\frac{\sum_{k=1}^{\tau} n^{m}_{k,\tau}}{\sum_{k=1}^{\tau} n^{m+1}_{k,\tau}} \right) \tag{5}$$

The RCMSE value is typically denoted as the sample entropy of sample en. for short. As with other techniques, the RCMSE curves are used to compare the relative complexity of normalized time series [10]. However, an advantage of the RCMSE method is that it has a lower chance of inducing the undefined entropy, as compared to earlier algorithms [9]. As was done in previous studies [8–10], the sample entropy, was plotted for scale-factor values ranging from 1 to 20.

3. Modeling and analysis

3.1. Weierstrass functions

Weierstrass functions are an example of a function, which is continuous but differentiable nowhere [31]. A proof of the non-differentiability of this function can be found in [32], and a discussion as to its fractal nature can be read in [33]. Typically, the Weierstrass function has a similar form to the following [34]:

$$W(t) = \sum_{k=1}^{\infty} \frac{e^{i(\gamma^k t + \varphi_k)}}{\gamma^{(2-D)k}} \tag{6}$$

where D is the fractional dimension with $1 < D < 2$, $\gamma > 1$, and φ_k is an arbitrary phase. Here, the real and imaginary parts of Eq. (6) are known as the Weierstrass cosine and sine functions, respectively. Additionally, D will be termed as the fractional dimension to avoid technical arguments over which type of dimension, D, represents, such as the box-counting dimension, fractal dimension, or the Hausdorff-Besicovitch dimension [34].

Although Weierstrass functions cannot be differentiated in the conventional sense, they have been shown to be differentiable to fractional order [35–39]. Furthermore, both integrating and differentiating functions to arbitrary order involve more generalized definitions, as compared to those found in the integer order calculus. For example, the fractional integral has been defined as [40–42]:

$${}_cD^{-\alpha}_t f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \left(t - t'\right)^{\alpha-1} f\left(t'\right) dt' \quad Re\,\alpha > 0 \tag{7}$$

Here Γ is the well-known gamma function, and α is the order of the derivative, which extends across the positive reals. Expanding upon Eq. (7), Oldham and Spanier show that the fractional derivative of a function, $f(t)$, may be written as [43,44]:

$$\begin{aligned} {}_aD^{\alpha}_t f(t) &= \frac{d^n}{dt^n} D^{\alpha-n}_t {}_a f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \left(t - t'\right)^{n-\alpha-1} f\left(t'\right) dt' \quad Re\,\alpha > 0 \end{aligned} \tag{8}$$

In the spirit of the work found in [35], we take the fractional integral, as defined in Eq. (7) and apply it to the righthand side (r. h. s.) of Eq. (6), while taking the limit of $c \to -\infty$ (from Eq. (7)):