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A gentle introduction to anisotropic banach spaces☆

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ABSTRACT

The use of anisotropic Banach spaces has provided a wealth of new results in the study of hyperbolic dynamical systems in recent years, yet their application to specific systems is often technical and difficult to access. The purpose of this note is to provide an introduction to the use of these spaces in the study of hyperbolic maps and to highlight the important elements and how they work together. This is done via a concrete example of a family of dissipative Baker's transformations. Along the way, we prove an original result connecting such transformations with expanding maps via a continuous family of transfer operators acting on a single Banach space.

1. Introduction

The study of anisotropic Banach spaces on which the transfer operator associated with a hyperbolic dynamical system has good spectral properties has been the subject of intense activity during the past 15 years. Beginning with the seminal paper [13], there have been a flurry of papers developing several distinct approaches, first for smooth uniformly hyperbolic maps [3,11,33,34], then piecewise hyperbolic maps [8,9,18], and finally to hyperbolic maps with more general singularities, including many classes of billiards [20,22] and their perturbations [21]. This technique has also been successfully applied to prove exponential rates of mixing for hyperbolic flows, a notoriously difficult problem, following a similar trajectory: first to contact Anosov flows [41,50], then to contact flows with discontinuities [10] and finally to billiard flows [7].

The purpose of this note is to provide a gentle introduction to the study of anisotropic Banach spaces via a concrete model: a family of dissipative Baker's transformations. This family of maps provides a prototypical hyperbolic setting and allows for the application of transfer operator techniques without the technical difficulties associated with other concrete models, such as dispersing billiards. On the other hand, it avoids the full generality necessary for an axiomatic treatment of Anosov or Axiom A maps as found, for example, in [11,34]. Despite its simplicity, the study of this family of maps includes all the essential elements needed for the successful application of this technique to more complex systems: a suitable set of norms, the Lasota–Yorke inequalities required to prove quasi-compactness of the transfer operator, a Perron–Frobenius argument for characterizing the peripheral spectrum, and the approximation of distributions in the Banach space norm. Thus we hope it will serve as an easily accessible introduction to the subject for those who wish to pursue this mode of analysis in more complex systems. This note is based, in part, on introductory lectures given at the June 2015 workshop DinAmici IV held in Corinaldo, Italy.

1.1. A brief survey of anisotropic spaces

As mentioned above, there has been a wealth of activity in the application of anistropic Banach spaces to the study of hyperbolic systems. Although they vary in their application, they have one feature in common: they all exploit the fact that in hyperbolic systems, the transfer operator improves the regularity of densities along unstable manifolds, and its dual improves the regularity of test functions along stable manifolds. This differentiated treatment of stable and unstable directions is what makes these spaces anisotropic.

In this section, we outline some of the principal branches of these efforts. Roughly, they can be divided into three groups:

(1) **The geometric approach pioneered by Liverani and Gouezel** [33]. This approach follows from the seminal work mentioned above by Blank, Keller and Liverani [13], who viewed the transfer operator as acting on distributions integrated against vector fields and various classes of test functions. An essential difference introduced in [33] was to integrate against smooth test functions on stable manifolds only, or more generally *admissible stable curves* whose tangent vectors lie in the stable cone, as opposed to integrating over the entire phase space, thus simplifying the application of the method. In the smooth case, this technique was able

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to exploit higher smoothness of the map to obtain improved bounds on the essential spectral radius of the transfer operator [33], and is applied to generalized potentials in [34]. It has also yielded significant results on dynamical determinants and zeta functions for both maps [42,44] and flows [31].

For systems with discontinuities, integrating on stable curves greatly simplifies the geometric arguments required to control the growth in complexity due to discontinuities. This was implemented first for two-dimensional piecewise hyperbolic maps (with bounded derivatives) [18], and then for various classes of billiards [20,22] and their perturbations [21]. It has also led to the recent proof of exponential mixing for some billiard flows [7]. This geometric approach has proved to be the most flexible so far in terms of the types of systems studied.

(2) **The Triebel-type spaces introduced by Baladi** [3]. These spaces are based on the use of Fourier transforms to convert derivatives into multiplication operators. They exploit the hyperbolicity of the map by taking negative fractional derivatives in the stable direction and positive derivatives in the unstable direction. Initially, the coordinates for these operators were tied to the invariant dynamical foliations associated with the hyperbolic systems, and these were assumed to be C^∞ [3]. They were later generalized to include more general families of foliations (only $C^{1+\epsilon}$ smooth and such that the family of foliations is invariant, not each foliation individually), and successfully applied to piecewise hyperbolic maps [8,9]. They were also the first norms succesfully adapted to contact flows with discontinuities in [10].

(3) **The Sobolev-type "microlocal spaces" of Baladi and Tsujii** [11]. In some sense, these spaces are an evolution of the Triebel-type spaces described above. The spaces still exploit the hyperbolicity of the map by using Fourier transforms and pseudo-differential operators, taking negative derivatives in the stable direction and positive derivatives in the unstable direction. Now, however, the invariant foliations are replaced by cones in the cotangent space on which these operators act, and the operators are averaged with respect to an L^p norm.

Such spaces, and the semi-classical versions of Faure and coauthors [25], have produced extremely strong results characterizing the spectrum of the transfer operator for smooth hyperbolic maps and flows [26–28]. This approach has also achieved the sharpest bound to date for the essential spectral radius of the transfer operator via a variational formula [12], and numerous results on dynamical determinants and zeta functions [6]. However, they have not been applied to systems with discontinuities, and it seems some new ideas will be needed to generalize them in this direction. Reference [5] contains a recent attempt to develop this capability.

This is only a brief description of the types of Banach spaces used to study hyperbolic systems in recent years, and is by no means a comprehensive listing. A more thorough and nuanced account is contained in the recent article of Baladi [4]. In addition, there are alternative approaches that use similar types of anisotropic constructions adapted to special cases. The recent work [30], for example, constructs spaces using an averaged type of bounded variation, which is shown to be effective for a class of partially hyperbolic maps with a skew-product structure.

In the present paper, we will follow the geometric approach of Liverani and Gouëzel described in (1) above. These spaces are the most concrete of the types listed above, the integrals being taken on stable manifolds against suitable test functions, and as such fit our purpose here best, which is to provide a hands-on introduction to the subject with as few pre-requisities as possible.

1.2. A pedagogical example

Before introducing the class of hyperbolic maps for which we will construct an appropriate anisotropic Banach space, we consider the following simpler example[1] of a contracting map of the interval.

For expanding systems, the fact that the transfer operator increases the regularity of densities is well-understood. It is this feature which generally enables one to derive the Lasota–Yorke inequalities needed to prove its quasi-compactness on a suitable space of functions compactly embedded in L^1 with respect to some reference measure. Although L^1 is in general both too small and too large a space in the context of hyperbolic maps, it is instructive to see that similar inequalities can be derived in the purely contracting case as well, and to note that the transfer operator in this case increases the regularity of *distributions*.

Let $I = [0, 1]$, and T: $I \circlearrowleft$ be a C^1 map satisfying $|T'(x)| \le \lambda < 1$ for all $x \in I$. It is a well-known consequence of the contraction mapping theorem that there exists a unique $a \in I$ such that $T(a) = a$.

For $\alpha \in (0, 1]$, let C^α denote the set of Hölder continuous functions on I with exponent α. For $\varphi \in C^\alpha$, define

$$H^\alpha(\varphi) = \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha}, \qquad \text{and } |\varphi|_{C^\alpha} = |\varphi|_{C^0} + H^\alpha(\varphi), \tag{1.1}$$

where $|\varphi|_{C^0} = \sup_{x \in I} |\varphi(x)|$. Since T is C^1, if $\varphi \in C^\alpha$, then $\varphi \circ T \in C^\alpha$.

Let $(C^\alpha)^*$ be the dual of C^α. For a distribution $\mu \in (C^\alpha)^*$, we define the action of the transfer operator $\mathcal{L}$ associated with T via its dual,

$$\mathcal{L}\mu(\varphi) = \mu(\varphi \circ T), \qquad \text{for all } \varphi \in C^\alpha.$$

Thus $\mathcal{L}\mu \in (C^\alpha)^*$ as well.

For $\mu \in (C^\alpha)^*$, define

$$\|\mu\|_\alpha = \sup_{\substack{\varphi \in C^\alpha \\ |\varphi|_{C^\alpha} \le 1}} \mu(\varphi).$$

The reader can check that $\|\cdot\|_\alpha$ satisfies the triangle inequality, and is a norm.

If $f \in C^1(I)$, then we can identify f with the measure $d\mu = f dm$, where m denotes Lebesgue measure on I. With this identification, $C^1 \subset (C^\alpha)^*$. When we regard $f \in C^1$ as an element of $(C^\alpha)^*$, we will write $f(\varphi) = \int_I f \varphi \, dm$.

We define $\mathcal{B}^\alpha$ to be the space $(C^\alpha)^*$ equipped with the $\|\cdot\|_\alpha$ norm. Note that $\mathcal{B}^\alpha$ is a Banach space of distributions that includes all Borel measures on I; this includes the point mass at a, δ_a.

We will work with the spaces $(\mathcal{B}^\alpha, \|\cdot\|_\alpha)$ and $(\mathcal{B}^1, \|\cdot\|_1)$, the latter of which has the same definitions as $\mathcal{B}^\alpha$, but with $\alpha = 1$.

For $\varphi \in C^\alpha$, $n \ge 0$, define $\overline{\varphi}_n = \int_I \varphi \circ T^n \, dm$, recalling that m denotes Lebesgue measure.

Now fix $\alpha < 1$. For $\varphi \in C^\alpha$ and $n \ge 0$, we estimate for $\mu \in \mathcal{B}^\alpha$,

$$\begin{aligned} \mathcal{L}^n\mu(\varphi) &= \mu(\varphi \circ T^n - \overline{\varphi}_n) + \mu(\overline{\varphi}_n) \\ &\le \|\mu\|_\alpha |\varphi \circ T^n - \overline{\varphi}_n|_{C^\alpha} + \|\mu\|_1 |\overline{\varphi}_n|_{C^1}. \end{aligned} \tag{1.2}$$

Since $\varphi \circ T^n \in C^\alpha$, there exists $u \in I$ such that $\varphi \circ T^n(u) = \overline{\varphi}_n$. Thus for $x, y \in I$, we have

$$\begin{aligned} &|\varphi \circ T^n(x) - \overline{\varphi}_n| = |\varphi \circ T^n(x) - \varphi \circ T^n(u)| \\ &\quad \le |\varphi|_{C^\alpha} |T^n(x) - T^n(u)|^\alpha \le \lambda^{n\alpha} |\varphi|_{C^\alpha}, \\ &\quad |\varphi \circ T^n(x) - \overline{\varphi}_n - \varphi \circ T^n(y) + \overline{\varphi}_n| \le |\varphi|_{C^\alpha} \lambda^{n\alpha} |x-y|^\alpha. \end{aligned} \tag{1.3}$$

[1] This example was communicated to the author by C. Liverani some years ago and has served as inspiration ever since.