Chaos, Solitons and Fractals 116 (2018) 14–21

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

New numerical method and application to Keller-Segel model with fractional order derivative

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ARTICLE INFO

Article history:
Received 11 August 2018
Revised 10 September 2018
Accepted 10 September 2018

Keywords:
Fractional calculus
Fundamental theorem of fractional calculus
Lagrange polynomial
Keller-Segel model

ABSTRACT

Using the fundamental theorem of fractional calculus together with the well-known Lagrange polynomial interpolation, we constructed a new numerical scheme. The new numerical scheme is suggested to solve non-linear and linear partial differential equation with fractional order derivative. The method was used to solve numerically the time fractional Keller-Segel model. The existence and uniqueness solution of the model with fractional Mittag-Leffler kernel derivative are presented in detail. Some simulations are performed to access the efficiency of the newly proposed method.

1. Introduction

In the last past years, several lives have been taken away due to cancer diseases. No wonder therefore why scientists in several fields are devoted to find a suitable to stop such fatal disease amount mankind. An effort is being doing by chemists to understanding the chemical reaction of this disease while mathematicians are interested in understand the dynamical system underpinning the spread of this disease. One-way to understanding this, mathematicians rely on ordinary differential and partial differential equations to build a mathematical able to replicate the observed facts. One of the first pioneers of this model is perhaps the Lee Aaron Segel who suggested the model of chemo taxis. For those do not know the scientific mean of cancer, we recall that, it is a group of diseases connecting nonconventional cell growth with a possibility of invading or spreading to other parts of the body for clear understanding of this process, we request readers to read through the following references [21–25]. Also we recall that, chemo taxis is a scientific word that is an addition of two others including chemo and taxis, meaning the diffusion of an organism in response to chemical stimulus more details please visit the following work [26–29]. Biologically speaking, somatic cells bacteria and other single/multicellular beings undeviating their diffusion accordingly to some chemical within their environment. It has been reported that, the mechanisms that tolerate chemo taxis in animals can also be destabilised during cancer metastasis. We note that, positive chemo taxis is observed if the diffusion is toward a higher concentration of the mentioned chemical, while on the other hand negative chemo taxis is identified when the movement is in the opposite direction. Using a capillary tube assay one can model the process of chemo taxis. With their biological instinct, motile prokaryotes are able to sense chemicals in their vicinities then, alter their motility consequently. The randomness is observed when no chemicals are present and the repellent chemical is present, the motility also change, runs become longer and tumbles because less frequent so that diffusion in the direction or in opposite direction to the chemical can be reached. Thus, the net diffusion can be considered. Thus, the diffusion can be considered in the beaker, where bacteria amass around the attractant and away from the repellent.

The concept of anomalous diffusion cannot be captured with classical differential operator as have been indicated in several already published studies. This is due to the fact that, the process of the spread does not follow a conventional pattern, and such pattern cannot be capturing using the concept of rate of change. It was suggested that only nonconventional or non-local differential operators are better candidates to capture such processes. However, due to the change in states during the diffusion of cancer within a human body, even those non-local differential operators based on power-law decay kernel cannot as they do not have ability if crossover either in waiting time distribution, mean square displacement and also density probability. Thus one will therefore rely on those differential operators with crossover behaviour and the better candidates are nothing more than the Atangana-

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https://doi.org/10.1016/j.chaos.2018.09.013
0960-0779/

Baleanu fractional differential operators [27,28]. Noting that, the mathematical model able to replicate this spread as suggested by Segel is highly non-linear, the model can only be solved using either numerical methods or iterative methods. Within the theoretical framework of the applications of these differential operators to modeling real world problem, some interesting results have been obtained and these results are very important when dealing with applications, we can quote the following works [30–42].

Within the passed years, many iterative methods have been introduced in order to take care of nonlinear partial differential equation with integer and non-integer order derivatives [1–6]. However, the problems faced by these methods are perhaps, the stability, the convergency and the ability of these methods to handle strong non-linearities. Methods like homotopy perturbation method, homotopy decomposition method, variational iterative method; Laplace/Sumudu perturbation method and Adomian decomposition method have been used and applied extensively within the passed decade [7–10]. Nevertheless, all these methods are not satisfactory due to their inabilities to secure the stability and the convergency for non-linear equations. Their limitations can be visible in the field of chaos as small change in the initial input can affect greatly the outputs. On the other hand a very powerful mathematical tools known as Adams-Bashforth method for solving nonlinear ordinary differential equations has been introduced, this method is very efficient and converges toward exact solution must of the times [11–15]. But this method has some limitations it is only application for ordinary differential equations and in addition, one may one to add the predictor-corrector to insure accuracy. However, the method was recognized as powerful numerical method for solving chaotic's, epidemiologic, and nonlinear models that arise in biology, chemistry, mechanic, technology and engineering. Thus there is a need of such numerical scheme for solving non-linear partial differential equations with no requirement corrector-predictor method. The new method will therefore be used to solve numerically the Keller-Segel model with Atangana-Baleanu fractional differentiation in Caputo sense.

2. Existence and uniqueness of Keller-Segel model with Atangana-Baleanu derivative

In this section the Keller-Segel model describing the aggregation advancement of cellular slime mold by chemical attraction. The mathematical model has been studied by several authors within the scope of classical differentiation and also fractional differentiation but with Riemann-Liouville and Caputo derivative. However, recently due the limitations of the power law kernel used in Riemann-Liouville derivative, Caputo and Fabrizio [16–20] suggested a different kernel, exponential decay law, this kernel although not non-local but has the property that it portrays fading memory which is observed in several physical problems in nature. To solve the problem on non-locality, Atangana and Baleanu suggested [19] new fractional derivatives with Mittag-leffler kernel. In this section we consider the following fractional non-linear model.

$$\begin{cases} {}_0^{ABC}D_t^{\alpha}u(x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \eta(u(x,t))}{\partial x}\right), \\ {}_0^{ABC}D_t^{\alpha}\rho(x,t) = b\frac{\partial^2 \rho(x,t)}{\partial x^2} + cu(x,t) - d\rho(x,t), \\ u(x,0) = f(x), \ \ \rho(x,0) = g(x), \\ {}_a^{ABC}D_t^{\alpha}u(x,t) = \frac{AB(\alpha)}{1-\alpha}\int\limits_0^t \frac{\partial u(x,l)}{\partial l}E_{\alpha}\left[-\alpha\frac{(t-l)^{\alpha}}{1-\alpha}\right]dl. \end{cases} \tag{1}$$

We shall present the detailed proof of existence and uniqueness of the exact solution using the fixed-point method.

2.1. Existence and uniqueness

To achieve the existence and uniqueness of exacts solution of Eq. (1), we apply on both sides the fractional integral, thus by the fundamental theorem of fractional calculus, we obtain:

$$\begin{cases} u(x,t) - u(x,0) = \frac{1-\alpha}{AB(\alpha)}\left(a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \eta(u(x,t))}{\partial x}\right)\right) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t \left(a\frac{\partial^2 u(x,l)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \eta(u(x,l))}{\partial x}\right)\right)(t-l)^{\alpha-1}dl, \\ \rho(x,t) - \rho(x,0) = \frac{1-\alpha}{AB(\alpha)}\int\limits_0^t b\frac{\partial^2 \rho(x,t)}{\partial x^2} + cu(x,t) - d\rho(x,t) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t b\frac{\partial^2 \rho(x,l)}{\partial x^2} + cu(x,l) - d\rho(x,l)(t-l)^{\alpha-1}dl. \end{cases} \tag{2}$$

For the sake of simplicity, we put

$$\begin{cases} \pi_1(x,t,u) = a\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \eta(u(x,t))}{\partial x}\right) \\ \pi_2(x,t,\rho) = b\frac{\partial^2 \rho(x,t)}{\partial x^2} + cu(x,t) - d\rho(x,t) \end{cases} \tag{3}$$

Thus Eq. (2) become

$$\begin{cases} u(x,t) - u(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_1(x,t,u) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \\ \qquad \int_0^t \pi_1(x,l,u)(t-l)^{\alpha-1}dl, \\ \rho(x,t) - \rho(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_2(x,t,\rho) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \\ \qquad \int_0^t \pi_2(x,l,\rho)(t-l)^{\alpha-1}dl. \end{cases} \tag{4}$$

To start we consider the infinite norm defines and also

$$\|\psi(t)\|_{\infty} = \sup_{t\in I}|\psi(t)| \tag{5}$$

$$M_1 = \max\left\{\pi_1(x,t,u)_{(x,t)\in I\times J}\right\}$$

$$M_2 = \max\left\{\pi_2(x,t,\rho)_{(x,t)\in I\times J}\right\}$$

We now consider the following fractional operators

$$\begin{cases} N\Phi - u(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_1(x,t,\Phi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_1(x,l,\Phi)(t-l)^{\alpha-1}dl, \\ \Delta\Xi - \rho(x,0) = \frac{1-\alpha}{AB(\alpha)}\pi_2(x,t,\Xi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_2(x,l,\Xi)(t-l)^{\alpha-1}dl. \end{cases} \tag{6}$$

We show that the above fractional operators are well defined; to achieve this we apply the infinite norm

$$\begin{cases} \|N\Phi - u(x,0)\|_{\infty} = \left\| \begin{matrix} \frac{1-\alpha}{AB(\alpha)}\pi_1(x,t,\Phi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_1(x,l,\Phi)(t-l)^{\alpha-1}dl \end{matrix} \right\|_{\infty}, \\ \|\Delta\Xi - \rho(x,0)\|_{\infty} = \left\| \begin{matrix} \frac{1-\alpha}{AB(\alpha)}\pi_2(x,t,\Xi) \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t \pi_2(x,l,\Xi)(t-l)^{\alpha-1}dl \end{matrix} \right\|_{\infty}. \end{cases}$$

$$\begin{cases} \|N\Phi - u(x,0)\|_{\infty} \leq \frac{1-\alpha}{AB(\alpha)}\|\pi_1(x,t,\Phi)\|_{\infty} \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\left\|\int\limits_0^t \pi_1(x,l,\Phi)(t-l)^{\alpha-1}dl\right\|_{\infty}, \\ \|\Delta\Xi - \rho(x,0)\|_{\infty} \leq \frac{1-\alpha}{AB(\alpha)}\|\pi_2(x,t,\Xi)\|_{\infty} \\ +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\left\|\int\limits_0^t \pi_2(x,l,\Xi)(t-l)^{\alpha-1}dl\right\|_{\infty}. \end{cases} \tag{7}$$

Thus using the max of the defined kernel, we obtained

$$\begin{cases} \|N\Phi - u(x,0)\|_{\infty} \leq \frac{1-\alpha}{AB(\alpha)}M_1 + \frac{M_1 T_{\max}^{\beta}}{AB(\alpha)\Gamma(\alpha)}, \\ \|\Delta\Xi - \rho(x,0)\|_{\infty} \leq \frac{1-\alpha}{AB(\alpha)}M_2 + \frac{M_2 T_{\max}^{\beta}}{AB(\alpha)\Gamma(\alpha)} + . \end{cases} \tag{8}$$

We define b the upper bound of I and $T_{\max}$ to be the upper bound of J, thus the functions are well defined if the following inequalities