Chaos, Solitons and Fractals 116 (2018) 114–120

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Frontiers

Modeling the dynamics of nutrient–phytoplankton–zooplankton system with variable-order fractional derivatives

Behzad Ghanbari[a], J.F. Gómez-Aguilar[b,*]

[a] Department of Engineering Science, Kermanshah University of Technology, Kermanshah, Iran

[b] CONACyT-Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N, Col. Palmira, C.P. 62490, Cuernavaca Morelos, México

A R T I C L E I N F O

Article history:
Received 31 August 2018
Revised 13 September 2018
Accepted 13 September 2018

Keywords:
Fractional calculus
Variable-order fractional derivatives
Nutrient–phytoplankton–zooplankton model
Lagrange interpolation

A B S T R A C T

We extended the nutrient–phytoplankton–zooplankton model involving variable-order fractional differential operators of Liouville–Caputo, Caputo–Fabrizio and Atangana–Baleanu. Variable-order fractional operators permits model and describe accurately real world problems, for example, diffusion or spread of nutrients or species in different states. Particularly, we model the interaction of nutrient phytoplankton and its predator zooplankton. The variable-order fractional numerical scheme based on the fundamental theorem of fractional calculus and the Lagrange polynomial interpolation was consider. Numerical simulation results are provided for illustrating the effectiveness and applicability of the algorithm to solve variable-order fractional differential equations.

1. Introduction

Fractional derivatives become excellent instrument for the description of memory and hereditary properties of various materials and processes. Such effects are in fact neglected in models with classical integer-order. This can be viewed as the main advantage of fractional derivatives. It also plays a crucial role in the description of dynamics between two different points in many other fields [1–11].

Despite of the idea of fractional derivatives and integrals can be considered as a generalization of corresponding standard ones, it is still quite a strange topic, very hard to explain. Because, unlike commonly used differential operators, it is not related to some important geometrical meaning, such as the trend of functions or their convexity. So, sometimes this mathematical tool could be judged “far from reality”. But indeed many physical phenomena have “intrinsic” fractional order description and so fractional order calculus is necessary in order to explain them [1]. There are a several number of definitions of fractional derivatives. For instance, Riemann and Liouville introduced the concept of fractional-order differentiation with power-law in [2,3]. Caputo and Fabrizio in [4], introduced a new derivative with fractional order based on the exponential-law and Atangana and Baleanu suggested another version of fractional-order derivative which uses the generalized Mittag-Leffler function with strong memory as non-local and non-singular kernel in [5].

In most cases, exact solutions of differential equations with integer, non-integer order or variable-order derivatives are very difficult to obtain; this is the principal motivation to develop iterative methods or numerical techniques to solve these equations. For the iterative methods (Adomian decomposition method, the variational iteration method, fractional sub-equation method, homotopy perturbation techniques, among others) [12–17], the principal problem are the stability and the convergence. Traditionally, Adams–Bashforth method has been recognized as a great and powerful numerical method able to provide a numerical solution of fractional differential equations [18–25]. In [26], the authors approximated Liouville–Caputo fractional derivatives by Chebyshev polynomials. Rosenfeld and Dixon in [27] developed a numerical scheme based on scattered data interpolation via reproducing kernel Hilbert spaces to solved Liouville–Caputo fractional order differential equations. In [28], the authors proposed a new three-step fractional Adams–Bashforth scheme for solving linear and nonlinear fractional order differential equations involving the Caputo–Fabrizio operator. Shahbazi and Javidi considered 3/8 Simpsons rule to design a new high order predictor-corrector scheme [29]. Modifications combining the rectangle formula, trapezoid formula, polynomial interpolation or Gauss–Lobatto quadrature can be found in [30,31]. In [32-33], the authors developed a generalized version of Adams–Basforth method to partial differential equations involving Laplace transform, Lagrange polynomial interpolation and the forward-backward scheme. Recently, in [34–36], the authors developed a constant-order and variable-order numerical schemes that combines the fundamental theorem of fractional calculus and the two-step Lagrange polynomial.

* Corresponding author.
E-mail address: jgomez@cenidet.edu.mx (J.F. Gómez-Aguilar).

https://doi.org/10.1016/j.chaos.2018.09.026
0960-0779/

In the biological models, interaction networks can be visualized as food-chains of species linked by trophic interactions. Phytoplankton provide food for marine life, oxygen for human being and purify the atmosphere by consuming carbon dioxide. Nevertheless, the rapid growth of phytoplankton may reduce the required amount of oxygen needed for the growth of other aquatic plants and animals. In the literature we found mathematical models to consider the interaction nutrient–phytoplankton–zooplankton. A mathematical model that describes three species food chain model consisting of toxin producing phytoplankton, zooplankton and fish population has been developed in [37]. In [38], the authors described a nutrient phytoplankton model by a couple of reaction-diffusion equations with delay. Biological systems presents long-range temporal memory or long-range space interactions, for this reason, the use of fractional derivatives can handle efficiently the dynamics of a disease model and also gives information on each point of the model. In [11] a fractional mathematical model for the interaction of nutrient phytoplankton and its predator zooplankton was investigated numerically. The fractional derivative of Liouville–Caputo type was used to obtain the generalization of the model. For solving the result fractional equations, a new numerical algorithm based on the polynomial interpolation was proposed.

In this paper, we consider a variable-order fractional nutrient–phytoplankton–zooplankton system [11] via Liouville–Caputo, Caputo–Fabrizio–Caputo and Atangana–Baleanu–Caputo fractional derivatives.

2. Mathematical model

The fractional nutrient–phytoplankton–zooplankton system [11] is generalized by replacing the classical derivative by the operator ${}_0\mathscr{D}_t^{\alpha(t)}$

$$ {}_0\mathscr{D}_t^{\alpha(t)}x_1(t) = \alpha_0 - ax_1(t) - b_1x_1(t)x_2(t) + c_1x_2(t) + c_2x_3(t), \tag{1a}$$

$$ {}_0\mathscr{D}_t^{\alpha(t)}x_2(t) = b_2x_1(t)x_2(t) - c_3x_2(t) - \frac{d_1x_2(t)x_3(t)}{e+x_2(t)}, \tag{1b}$$

$$ {}_0\mathscr{D}_t^{\alpha(t)}x_3(t) = \frac{d_2x_2(t)x_3(t)}{e+x_2(t)} - fx_2(t)x_3(t) - c_4x_3(t), \tag{1c}$$

with initial conditions

$$x_1(0) = x_{1,0} > 0, \qquad x_2(0) = x_{2,0} > 0, \qquad x_3(0) = x_{3,0} > 0,$$

where α_0, a, b_1, b_2, c_1, c_2, c_3, c_4, d_1, d_2, e and f are positive constants.

In the above model $x_1(t)$ denote the concentration of nutrient, $x_2(t)$ denotes the biomass of phytoplankton which also produces toxicant harmful to the zooplankton biomass and $x_3(t)$ denote the concentration of zooplankton population, the parameters $\{a, b_1, b_2, c_1, c_2, c_3, c_4, d_1, d_2, e, f, \alpha_0\}$ represents the rate of nutrient loss; nutrient uptake rate for the phytoplankton population; nutrient-phytoplankton conversion rate; nutrient recycling rate after the death of phytoplankton; nutrient recycling rate after the death of zooplankton; phytoplankton mortality rate; zooplankton death rate, maximal zooplankton ingestion rate; maximal phytoplankton–zooplankton conversion rate; half saturation constant for a Holling type II functional response; rate of zooplankton decay due to toxin producing phytoplankton and the constant input nutrient concentration, respectively. More details of this model can be found in [11].

The variable-order operator ${}_0\mathscr{D}_t^{\alpha(t)}$ can be of type ${}_0^C\mathscr{D}_t^{\alpha(t)}$, ${}_0^{CFC}\mathscr{D}_t^{\alpha(t)}$ or ${}_0^{ABC}\mathscr{D}_t^{\alpha(t)}$, called Liouville–Caputo, Caputo–Fabrizio–Caputo or Atangana–Baleanu–Caputo fractional derivatives with variable-order $\alpha(t)$, respectively.

The variable-order Liouville–Caputo fractional derivative with power-law (C) is defined as follows [7]:

$$ {}_0^C\mathcal{D}_t^{\alpha(t)}f(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-\tau)^{-\alpha(t)}\dot{f}(\tau)d\tau, \quad 0<\alpha(t)\le 1. \tag{2}$$

The variable-order Caputo–Fabrizio fractional derivative with exponential-law in Liouville–Caputo sense (CFC) is defined as follows [36]:

$$ {}_0^{CFC}\mathcal{D}_t^{\alpha(t)}f(t) = \frac{(2-\alpha(t))M(\alpha(t))}{2(1-\alpha(t))}\int_0^t \times \exp\left[-\alpha(t)\frac{(t-\tau)}{1-\alpha(t)}\right]\dot{f}(\tau)d\tau, \quad 0<\alpha(t)<1, \tag{3}$$

where $M(\alpha(t)) = \frac{2}{2-\alpha(t)}$ is a normalization function.

The variable-order Atangana–Baleanu–Caputo fractional derivative with Mittag-Leffler (ABC) is defined as follows [5,6]

$$ {}_0^{ABC}\mathcal{D}_t^{\alpha(t)}f(t) = \frac{B(\alpha(t))}{1-\alpha(t)}\int_0^t E_{\alpha(t)}\left[-\alpha(t)\frac{(t-\tau)^{\alpha(t)}}{1-\alpha(t)}\right] \times \dot{f}(\tau)d\tau, \quad 0<\alpha(t)\le 1, \tag{4}$$

where $B(\alpha(t)) = 1-\alpha(t) + \dfrac{\alpha(t)}{\Gamma(\alpha(t))}$ is a normalization function.

Now considering the numerical scheme developed in [36], we obtain numerical simulations for the nutrient–phytoplankton–zooplankton model in Liouville–Caputo; Caputo–Fabrizio–Caputo; and Atangana–Baleanu–Caputo fractional derivatives with variable order $\alpha(t)$.

3. Numerical schemes

3.1. Numerical scheme in Liouville–Caputo sense with variable-order

A fractional ordinary differential equation of Liouville–Caputo type with variable-order can be expressed as follows:

$$ {}_0^C\mathscr{D}_t^{\alpha(t)}y(t) = f(t, y(t)). \tag{5}$$

The approximate solution of Eq. (5) is obtained as [36]

$$ y_{n+1}(t) = y(0) + \frac{1}{\Gamma(\alpha(t))}\sum_{m=0}^{n}\Bigg(\frac{h^{\alpha(t)}f(t_m,y_m)}{\alpha(t)(\alpha(t)+1)} ((n+1-m)^{\alpha(t)}(n-m+2+\alpha(t)) - (n-m)^{\alpha(t)} \times(n-m+2+2\alpha(t))) - \frac{h^{\alpha(t)}f(t_{m-1},y_{m-1})}{\alpha(t)(\alpha(t)+1)} \times((n+1-m)^{\alpha(t)+1} - (n-m)^{\alpha(t)}(n-m+1+\alpha(t)))\Bigg). \tag{6}$$

3.2. Numerical scheme in Caputo–Fabrizio–Caputo sense with variable-order

Now, we have the following fractional differential equation with variable-order in Caputo–Fabrizio–Caputo sense

$$ {}_0^{CFC}\mathscr{D}_t^{\alpha(t)}y(t) = f(t, y(t)). \tag{7}$$

The numerical solution fo Eq. (7) is obtained by the following expression [36]

$$ y_{n+1} = y_n + \left[\frac{(2-\alpha(t))(1-\alpha(t))}{2} + \frac{3h}{4}\alpha(t)(2-\alpha(t))\right]f(t_n,y_n) - \left[\frac{(2-\alpha(t))(1-\alpha(t))}{2} + \frac{h}{4}\alpha(t)(2-\alpha(t))\right]f(t_{n-1},y_{n-1}). \tag{8}$$