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Collocation methods for fractional differential equations involving non-singular kernel

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ABSTRACT

A system of fractional differential equations involving non-singular Mittag-Leffler kernel is considered. This system is transformed to a type of weakly singular integral equations in which the weak singular kernel is involved with both the unknown and known functions. The regularity and existence of its solution is studied. The collocation methods on discontinuous piecewise polynomial space are considered. The convergence and superconvergence properties of the introduced methods are derived on graded meshes. Numerical results provided to show that our theoretical convergence bounds are often sharp and the introduced methods are efficient. Some comparisons and applications are discussed.

1. Introduction

Increasing the use of fractional calculations has increased the variety of questions and, resulted in various basic definitions for fractional integral and derivative. We recall that the Riemann–Liouville definition entails physically unacceptable initial conditions [1]; conversely for the Caputo fractional derivative, the initial conditions are expressed in terms of integer-order derivatives having direct physical significance [1,2]. Few years ago, Caputo and Fabrizio have opened the following debate within the fractional community: is it possible to describe all non-local phenomena within the same basic kernels, namely the power kernel involved within the definition of Riemann-Liouville derivative and some other few basic fractional derivatives [3]. They proposed a non-local definition for fractional derivative. Immediately, Nieto and Losada found the associated integral of the Caputo-Fabrizio fractional operator [4]. In this respect, Atangana and Baleanu have introduced a fractional derivative with non-singular Mittag-Leffler kernel [5].

The importance and application of the Mittag-Leffler kernel for generalized diffusion, Fokker-Planck-Smoluchowski equations and generalized Langevin equation can be found in the works of [6–8]. The corresponding tempered Mittag-Leffler memory kernel with $b = 0$ and $\beta = 1$ turns to the one parameter Mittag-Leffler memory kernel. Another related non-singular integral operator is Prabhakar operator which its properties and applications can be found in [9–12].

The definition by Atangana and Baleanu was tested with success in many fields including chaotic behavior, epidemiology, thermal science, hydrology, mechanical engineering and biology [5,13–24].

The dynamics of many physical or biological problem can be modeled by a system of fractional differential equations (FDE). Using the new definition [5] (${}^{ABC}_{\ \ 0}D_t^{\alpha}$), a system of Mittag–Leffler non-singular FDEs (AB type FDEs) can be described by

$$\begin{aligned} {}^{ABC}_{\ \ 0}D_t^{\alpha} y(t) &= Ay(t) + f(t), \ \ t \in I := [0, T] \\ y(0) &= y_0, \end{aligned} \tag{1}$$

where A is a constant matrix of dimension $\nu \times \nu$, $n \in \mathcal{N}$ is the dimension of the system, $f : \mathcal{R} \to \mathcal{R}^{\nu}$ is a known vector-valued function, and $y : \mathcal{R} \to \mathcal{R}^{\nu}$ is the unknown function.

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Recently, it is observed that the system (1) is more successful for modeling of suspension concentration distribution in turbulent flows than other models [25].

The conditions for the existence and uniqueness of the solution to exponential non-singular system can be found in [26]. The consistency condition

$$Ay_0 + f(0) = 0,$$

is one of them. It seems that this condition is also important for system (1) with Mittag-Leffler non-singular kernels. This imposes some restriction on system (1). However, due to the important dynamics of the solutions of system (1), it is very important to solve the system (1) analytically or numerically [27].

One of the classical method for solving integral or differential equations is collocation methods on piecewise polynomial spaces [28,29]. Recently, these methods have received more attention, for fractional differential equations involving classical definitions of fractional derivative or integrals [30–34]. Here, we investigate these methods for system of fractional differential equations involving new definitions of fractional derivative with non-singular Mittag-Leffler kernel. The aim of this paper is to implement the collocation method on piecewise polynomial spaces for solving the system 1. The other purpose of the paper is to obtain convergence and superconvergence analysis for the proposed methods.

In this paper, first we use appropriate operator to transform the system 1 to a system with weakly singular integral operators. Since the integral equations with weakly singular kernels do not converge rapidly, we use the graded mesh to obtain higher order methods.

The structure of this paper is as follows. In Section 2, we provide a preliminary definitions related to Mittag-Leffler function and AB fractional derivative and transforming fractional differential equation to an integral equation with weak singular kernel. In Section 3, we introduce the discontinuous piecewise polynomial collocation methods for solving the system 1, in details. In Section 4, we obtain the regularity of the solutions. In Section 5, we study a convergence analysis for proposed methods. In Section 6, we obtain the super-convergence results. Finally, in Section 7, we provide some numerical examples to show the efficiency of the introduced method and to confirm the theoretical results. Also, we apply the method for solving diffusion equation and we compare the singular and the non-singular diffusion equation.

2. Definitions and preliminaries

In this section, we first recall some basic definitions and results related to the Mittag-Leffler function [35]. Then, we recall some basic definitions and results related to the new non-singular fractional derivative and integral formulae [5].

2.1. The Mittag-Leffler function

The Mittag-Leffler function is the cornerstone of the fractional calculus. Several books and excellent papers [35–37] describe the importance of these types of operators. The concept of Mittag-Leffler calculus was introduced in [5] and the integral associated to the non-singular fractional operator with Mittag-Leffler kernel was found by using the Laplace transform [24,38].

Throughout the paper, the symbol E_α shows the one parameter Mittag-Leffler function [39] defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad \mathrm{Re}(\alpha) > 0.$$

The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha, \beta \in \mathcal{C}, \mathrm{Re}(\alpha) > 0).$$

Theorem 1 *[40]. Let $\rho, \mu, \upsilon, \omega \in \mathcal{C}$, $(\mathrm{Re}(\rho), \mathrm{Re}(\mu), \mathrm{Re}(\upsilon) > 0)$.Then,*

$$\int_0^x (x-t)^{\mu-1} E_{\rho,\mu}(\omega(x-t)^\rho) t^{\upsilon-1} dt = \Gamma(\upsilon) x^{\mu+\upsilon-1} E_{\rho,\mu+\upsilon}(\omega x^\rho). \tag{2}$$

2.2. AB type non-singular fractional derivative and integral

We use a Sobolev space defined by

$$\mathcal{H}^1[t_0, t_f] := \left\{ u \in L^2[t_0, t_f] : \frac{du}{dt} \in L^2[t_0, t_f] \right\}$$

to define AB type fractional derivative as follow:

Definition 2. For $f \in \mathcal{H}^1[t_0, t_f]$ and $0 < \alpha < 1$, the (left) AB fractional derivative in the Riemann-Liouville sense is defined by ([5]):

$$^{ABR}_{\ \ t_0}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_{t_0}^t f(\tau) E_\alpha\left(-\alpha \frac{(t-\tau)^\alpha}{1-\alpha}\right) d\tau, \tag{3}$$

and in the Caputo sense is defined by

$$^{ABC}_{\ \ t_0}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_{t_0}^t \frac{df(\tau)}{d\tau} E_\alpha\left(-\alpha \frac{(t-\tau)^\alpha}{1-\alpha}\right) d\tau, \tag{4}$$

where $B(\alpha)$ is a normalization function obeying $B(0) = B(1) = 1$.

The associated fractional integral is also defined by [5]:

$$\begin{aligned} ^{AB}_{\ t_0}I_t^\alpha f(t) &= \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{t_0}^t f(\tau)(t-\tau)^{\alpha-1} d\tau \\ &= \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)}\ {}_{t_0}I_t^\alpha f(t). \end{aligned} \tag{5}$$

The fractional integral of $(x-t_0)^\beta$ is

$${}_{t_0}I_t^\alpha (t-t_0)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-t_0)^{\beta+\alpha}, \quad \beta > -1, \quad \alpha > 0.$$

2.3. Transforming fractional differential equation to an integral equation with weak singular kernel

The Newton-Leibniz formula for AB fractional derivative and integral is obtained in [38,41].

Proposition 3. *For $0 < \alpha < 1$, we have [38]:*

$$\left(^{AB}_{\ t_0}I_t^\alpha \ ^{ABC}_{\ \ t_0}D_t^\alpha \right) f(t) = f(t) - f(t_0). \tag{6}$$

From Theorem 1, we can obtain the AB fractional derivative of a monomial t^β, $(\beta > 0)$.

$$^{ABC}_{\ \ 0}D_t^\alpha t^\beta = \frac{B(\alpha)\Gamma(\beta+1)}{1-\alpha} t^\beta E_{\alpha,1+\beta}\left(-\frac{\alpha}{1-\alpha} t^\alpha\right), \quad \beta > 0, \quad \alpha > 0. \tag{7}$$

Taking fractional integration from both side of the system (1) and using (6), the system (1) can be written in the following form.

$$\begin{aligned} \left(\mathbf{I} - \frac{1-\alpha}{B(\alpha)} A\right) y(t) &= \frac{\alpha}{B(\alpha)} A I^\alpha y(t) + y_0 + \frac{1}{B(\alpha)} ((1-\alpha) f(t) \\ &\quad + \alpha I^\alpha f(t)). \end{aligned} \tag{8}$$

Let $E = \mathbf{I} - \frac{1-\alpha}{B(\alpha)} A$. Then, using following lemma, one can guarantee the invertibility of E.