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Two weight inequality for Bergman projection

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ABSTRACT

The motivation of this paper comes from the two weight inequality

$$\|P_\omega(f)\|_{L^p_v} \leq C\|f\|_{L^p_v}, \quad f \in L^p_v,$$

for the Bergman projection P_ω in the unit disc. We show that the boundedness of P_ω on L^p_v is characterized in terms of self-improving Muckenhoupt and Bekollé–Bonami type conditions when the radial weights v and ω admit certain smoothness. En route to the proof we describe the asymptotic behavior of the L^p-means and the L^p_v-integrability of the reproducing kernels of the weighted Bergman space A^2_ω.

RÉSUMÉ

La motivation de cet article vient de l'inégalité suivante

$$\|P_\omega(f)\|_{L^p_v} \leqslant C\|f\|_{L^p_v}, \quad f \in L^p_v,$$

où P_ω est la projection de Bergman sur le disque unité.
On démontre que la continuité de P_ω sur L^p_v est caractérisée en termes de conditions auto-améliorantes de type Muckenhoupt et Bekollé–Bonami quand les poids radiaux v et ω sont suffisamment réguliers.
En établissant notre résultat principal on décrit aussi le comportement asymptotique de la moyenne L^p et de l'integrabilité L^p_v des noyaux reproduisants des espaces Bergman A^2_ω.

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1. Introduction

Let A^2_ω denote the subspace of analytic functions in L^2_ω induced by a nonnegative integrable function ω on the unit disc $\mathbb{D}$. If the norm convergence in the Bergman space A^2_ω implies the uniform convergence on compact subsets, the Hilbert space A^2_ω is a closed subspace of L^2_ω and the orthogonal Bergman projection P_ω from L^2_ω to A^2_ω is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\,\omega(\zeta)dA(\zeta),$$

where B^ω_z are the reproducing kernels of A^2_ω.

In this paper we are mainly interested in the question of when

$$\|P_\omega(f)\|_{L^p_v} \le C\|f\|_{L^p_v}. \tag{1}$$

To the best of our knowledge, the existing literature does not offer an answer even in the case where $\omega = v$ is radial. The boundedness of projections on L^p-spaces is an intriguing topic which has attracted a considerable amount of attention during the last decades. This is not only due to the mathematical difficulties the question raises, but also to its numerous applications in operator theory. Recently, the bounded projections $P_0 : L^2_{|g|^{-2}} \to L^2_{|f|^2}$ were characterized on the way to disprove the Sarason conjecture on the Toeplitz product operator $T_f T^\star_g : A^2 \to A^2$, induced by analytic symbols f and g [2]. However, the most commonly known results concerning the two weight inequality (1) have been obtained when the inducing weight ω is standard [5,6]. In this case the reproducing kernels are given by the neat expression $(1-\overline{z}\zeta)^{-(2+\alpha)}$ that is easy to work with. The general situation is much more complicated because of the lack of explicit expressions for B^ω_z. Because of this fact, and due to previous studies [8,9,21] revealing the importance that the decay of the weight plays in the analysis of (1), we will focus on so-called regular and rapidly increasing (radial) weights. Postponing the exact definitions of these weights to the next section, we will denote these classes of weights by $\mathcal{R}$ (for regular) and $\mathcal{I}$ (for rapidly increasing).

The techniques employed here to study (1) require L^p-estimates for the Bergman reproducing kernels B^ω_z. The first of the main results describes the asymptotic behavior of the L^p-means of B^ω_z (or its derivatives). The latter part of this theorem reveals a precise estimate for the L^p_v-integral of B^ω_z. Needless to say that such kernel estimates are frequently applied in the operator theory.

The main result of this study characterizes those regular weights ω and v for which (1) holds. In particular we show that they coincide with those for which the sublinear operator

$$P^+_\omega(f)(z) = \int_{\mathbb{D}} |f(\zeta)||B^\omega_z(\zeta)|\,\omega(\zeta)dA(\zeta)$$

is bounded on L^p_v. The characterizing integral condition is equivalent, on one hand, to a Muckenhoupt-type condition related to Hardy operators [14], and on the other hand, to a generalization of the classical Bekollé–Bonami condition. In contrast to the general situation for Bekollé–Bonami weights [7], all these conditions are self-improving.

As a byproduct, we will show that P^+_ω is bounded on L^p_ω if $\omega \in \mathcal{R}$ and $p > 1$. The situation is different for $\omega \in \mathcal{I}$ because then P^+_ω is not bounded on L^p_ω. These results emphasize the general phenomena that many finer function-theoretic properties valid for A^p_α just simply break down for A^p_ω induced by $\omega \in \mathcal{I}$.

Throughout the paper $\frac{1}{p} + \frac{1}{p'} = 1$. Further, the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \le Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.