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Approximate current-vortex sheets near the onset of instability ☆

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A B S T R A C T

The paper is concerned with the free boundary problem for 2D current-vortex sheets in ideal incompressible magneto-hydrodynamics near the transition point between the linearized stability and instability. In order to study the dynamics of the discontinuity near the onset of the instability, Hunter and Thoo [9] have introduced an asymptotic quadratically nonlinear integro-differential equation for the amplitude of small perturbations of the planar discontinuity. We study such amplitude equation and prove its nonlinear well-posedness under a stability condition given in terms of a longitudinal strain of the fluid along the discontinuity.

R É S U M É

Dans cet article on traite un problème à frontière libre pour des solutions de type « current-vortex sheet » en 2D de la magnéto-hydrodynamique incompressible idéale au voisinage d'un point de transition entre la stabilité et l'instabilité linéarisées. Afin d'étudier la dynamique de la discontinuité au voisinage de l'apparition de l'instabilité, Hunter et Thoo [9] ont introduit une équation intégro-différentielle quadratique non linéaire asymptotique pour l'amplitude des petites perturbations de la discontinuité plane. On étudie une telle équation d'amplitude et on montre qu'elle est bien posée, sous une condition de stabilité donnée en termes de la déformation longitudinale du fluide le long de la discontinuité.

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1. Introduction and main results

We consider the equations of 2-dimensional incompressible magneto-hydrodynamics (MHD)

$$\begin{cases} \partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbf{B} \otimes \mathbf{B}) + \nabla q = 0\,, \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0\,, \\ \operatorname{div} \mathbf{u} = 0\,,\ \operatorname{div} \mathbf{B} = 0 \qquad\qquad \text{in } (0,T) \times \mathbb{R}^2, \end{cases} \tag{1}$$

where $\mathbf{u} = (u_1, u_2)$ denotes the velocity field and $\mathbf{B} = (B_1, B_2)$ the magnetic field, p is the pressure, $q = p + \frac{1}{2}|\mathbf{B}|^2$ the total pressure (for simplicity the density $\rho \equiv 1$).

Let us consider current-vortex sheets solutions of (1) (also called "tangential discontinuities"), that is weak solutions that are smooth on either side of a smooth hypersurface

$$\Gamma(t) = \{y = f(t,x)\}, \qquad \text{where } t \in [0,T],\ (x,y) \in \mathbb{R}^2,$$

and such that at $\Gamma(t)$ satisfy the boundary conditions

$$\partial_t f = \mathbf{u}^\pm \cdot N\,, \quad \mathbf{B}^\pm \cdot N = 0\,, \quad [q] = 0\,, \tag{2}$$

with $N := (-\partial_x f, 1)$. In (2) $(\mathbf{u}^\pm, \mathbf{B}^\pm, q^\pm)$ denote the values of $(\mathbf{u}, \mathbf{B}, q)$ on the two sides of $\Gamma(t)$, and $[q] = q^+_{|\Gamma} - q^-_{|\Gamma}$ the jump across $\Gamma(t)$.

From (2) the discontinuity front $\Gamma(t)$ is a tangential discontinuity, namely the plasma does not flow through the discontinuity front and the magnetic field is tangent to $\Gamma(t)$. The possible jump of the tangential velocity and tangential magnetic field gives a concentration of current and vorticity on the front $\Gamma(t)$. Current-vortex sheets are fundamental waves in MHD and play an important role in plasma physics and astrophysics. The existence of current-vortex sheets solutions is known for compressible fluids [6,19], but, as far as we know, is still an open problem for incompressible fluids, see [7,14] for partial results.

The necessary and sufficient linear stability condition for planar (constant coefficients) current-vortex sheets was found a long time ago, see [4,12,18]. To introduce it, let us consider a stationary solution of (1), (2) with interface located at $\{y = 0\}$ given by the constant states

$$\mathbf{u}^\pm = (U^\pm, 0)^T, \qquad \mathbf{B}^\pm = (B^\pm, 0)^T \tag{3}$$

in the x-direction. The necessary and sufficient stability condition for the stationary solution is

$$|U^+ - U^-|^2 < 2\left(|B^+|^2 + |B^-|^2\right), \tag{4}$$

see [4,12,18]. Equality in (4) corresponds to the transition to *violent* instability, i.e. ill-posedness of the linearized problem.

Let $U = (U^+, U^-), B = (B^+, B^-)$ and define

$$\Delta(U,B) := \frac{1}{2}\left(|B^+|^2 + |B^-|^2\right) - \frac{1}{4}|U^+ - U^-|^2.$$

According to (4), stability/instability occurs when $\Delta(U,B) \gtrless 0$.

Hunter and Thoo investigated in [9] the transition to instability when $\Delta(U,B) = 0$. Assume that $U^\pm, B^\pm$ depend on a small positive parameter ε and

$$U^\pm = U_0^\pm + \varepsilon U_1^\pm + O(\varepsilon^2), \qquad B^\pm = B_0^\pm + \varepsilon B_1^\pm + O(\varepsilon^2)$$