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# On the invertibility of Born–Jordan quantization

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A B S T R A C T

As a consequence of the Schwartz kernel Theorem, any linear continuous operator $\widehat{A}\colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ can be written in Weyl form in a unique way, namely it is the Weyl quantization of a unique symbol $a \in \mathcal{S}'(\mathbb{R}^{2n})$. Hence, dequantization can always be performed, and in a unique way. Despite the importance of this topic in Quantum Mechanics and Time-frequency Analysis, the same issue for the Born–Jordan quantization seems simply unexplored, except for the case of polynomial symbols, which we also review in detail. In this paper we show that any operator $\widehat{A}$ as above can be written in Born–Jordan form, although the representation is never unique if one allows general temperate distributions as symbols. Then we consider the same problem when the space of temperate distributions is replaced by the space of smooth slowly increasing functions which extend to entire function in $\mathbb{C}^{2n}$, with a growth at most exponential in the imaginary directions. We prove again the validity of such a representation, and we determine a sharp threshold for the exponential growth under which the representation is unique. We employ techniques from the theory of division of distributions.



R É S U M É

Comme conséquence du théorème des noyaux de Schwartz chaque opérateur linéaire continu $\widehat{A} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ peut être écrit d'une manière unique comme un opérateur de Weyl, c'est-à-dire comme la quantification de Weyl d'un symbole $a \in \mathcal{S}'(\mathbb{R}^{2n})$ unique. Il s'ensuit que la « déquantification » d'un tel opérateur peut toujours être effectuée, et ceci de manière unique. Malgré l'importance de ce résultat en mécanique quantique et en analyse temps-fréquence, cette question n'a pas été envisagée dans le cas de la quantification de Born et Jordan, sauf dans le cas des symboles polynomiaux, abordé ici. Dans cet article on montre que tout opérateur $\widehat{A}$ défini comme ci-dessus peut être écrit comme un opérateur de Born–Jordan, bien que cette représentation ne soit jamais unique si l'on considère des symboles suffisamment généraux (distributions tempérées). On étudie ensuite le même problème en remplaçant l'espace des distributions tempérées par l'espace des fonctions infiniment différentiables à croissance lente qui se prolongent en des

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fonctions entières sur $\mathbb{C}^n$ à croissance au plus polynomiale dans les directions imaginaires pures. On démontre encore une fois la validité d'une telle représentation, ainsi que l'existence d'un seuil précis pour la croissance exponentielle assurant l'unicité de cette représentation. On utilise pour cela des techniques empruntées à la théorie de la division des distributions.



## 1. Introduction

Roughly speaking, quantization is the process of associating to a function or distribution defined on phase space an operator. Historically, this notion appears explicitly for the first time in Born and Jordan's foundational paper [4] where they set out to give a firm mathematical basis to Heisenberg's matrix mechanics. Born and Jordan's quantization scheme was strictly speaking limited to polynomials in the variables $x$ and $p$; it was soon superseded by another rule due to Weyl, and whose extension is nowadays the preferred quantization in physics. However, it turns out that there is a recent regain in interest in an extension of Born and Jordan's initial rule, both in Quantum Physics and Time-frequency Analysis. In fact, on the one hand it is the correct rule if one wants matrix and wave mechanics to be equivalent quantum theories (see the discussion in [14]). On the other hand, as a time-frequency representation the Born–Jordan distribution has been proved to be surprisingly successful, because it allows to damp very well the unwanted "ghost frequencies", as shown in [2,36].

The difference between Born–Jordan and Weyl quantization is most easily apprehended on the level of monomial quantization: in dimension $n = 1$ for any integers $r, s \geq 0$ we have

$$\mathrm{Op_W}(x^r p^s) = \frac{1}{2^s}\sum_{\ell=0}^{s}\binom{s}{\ell}\widehat{p}^{\,s-\ell}\widehat{x}^{\,r}\widehat{p}^{\,\ell} = \frac{1}{2^r}\sum_{\ell=0}^{r}\binom{r}{\ell}\widehat{x}^{\ell}\widehat{p}^{\,s}\widehat{x}^{\,r-\ell} \tag{1}$$

(see [26]) and

$$\mathrm{Op_{BJ}}(x^r p^s) = \frac{1}{s+1}\sum_{\ell=0}^{s}\widehat{p}^{\,s-\ell}\widehat{x}^{\,r}\widehat{p}^{\,\ell} = \frac{1}{r+1}\sum_{\ell=0}^{r}\widehat{x}^{\ell}\widehat{p}^{\,s}\widehat{x}^{\,r-\ell} \tag{2}$$

(see [4]). As usual here $\widehat{p} = -i\hbar\partial/\partial x$ and $\widehat{x}$ is the multiplication operator by $x$. The Born–Jordan scheme thus appears as being an equally-weighted quantization, as opposed to the Weyl scheme: $\mathrm{Op_{BJ}}(x^r p^s)$ is the average of all possible permutations of the product $\widehat{x}^{\,r}\widehat{p}^{\,s}$.

One can extend the Weyl and Born–Jordan quantizations to arbitrary symbols $a \in \mathcal{S}'(\mathbb{R}^{2n})$ by defining the operators $\widehat{A}_{\mathrm{W}} = \mathrm{Op_W}(a)$ and $\widehat{A}_{\mathrm{BJ}} = \mathrm{Op_{BJ}}(a)$: $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ as

$$\begin{aligned}\widehat{A}_{\mathrm{W}}\psi &= \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z)\widehat{T}(z)\psi dz,\\ \widehat{A}_{\mathrm{BJ}}\psi &= \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z)\Theta(z)\widehat{T}(z)\psi dz,\end{aligned} \tag{3}$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)$ and the integrals are to be understood in the distributional sense; here $\widehat{T}(z_0) = e^{-i(x_0\widehat{p}-p_0\widehat{x})/\hbar}$, $z_0 = (x_0, p_0)$, is the Heisenberg operator, $a_\sigma(z) = a_\sigma(x,p) = Fa(p,-x)$, with $z = (x,p)$, is the symplectic Fourier transform of $a$, and $\Theta$ is Cohen's [6] kernel function, defined by

$$\Theta(z) = \frac{\sin(px/2\hbar)}{px/2\hbar}$$