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Morrey global bounds and quasilinear Riccati type equations below the natural exponent

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Abstract

We obtain global bounds in Lorentz–Morrey spaces for gradients of solutions to a class of quasilinear elliptic equations with low integrability data. The results are then applied to obtain sharp existence results in the framework of Morrey spaces for Riccati type equations with a gradient source term having growths below the natural exponent of the operator involved. A special feature of our results is that they hold under a very general assumption on the nonlinear structure, and under a mild natural restriction on the boundary of the ground domain.

Résumé

On obtient des bornes globales dans les espaces de Lorentz–Morrey sur le gradient des solutions d'une classe d'équations elliptiques quasilinéaires pour des données faiblement intégrables. Ces résultats sont ensuite utilisés pour obtenir l'existence de solutions dans des espaces de Morrey d'équations de Riccati sous une hypothèse de croissance du gradient du terme source inférieure à celle de l'exposant naturel de l'opérateur. Une particularité de ces résultats est qu'ils s'appliquent sous des hypothèses très générales sur la structure de la non-linéarité, et la frontière du domaine.

Keywords: Quasilinear elliptic operator; Morrey space; Uniformly thick domain; Riccati type equation

1. Introduction

There are two main goals that we wish to accomplish in this paper. The first goal is to obtain global regularity in Morrey and Lorentz–Morrey spaces for gradients of solutions to nonhomogeneous quasilinear equations of the form

$$\begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla u)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The second goal is to address the solvability of the following quasilinear Riccati type equation

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$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = |\nabla u|^q + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded open set of $\mathbb{R}^n$, $n \geqslant 2$, and for now the data f is a function in $L^1(\Omega)$ or a finite measures in Ω.

In (1.1) and (1.2) the nonlinearity $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector valued function, i.e., $\mathcal{A}(x, \xi)$ is measurable in x for every ξ and continuous in ξ for a.e. x. We assume that $\mathcal{A}$ satisfies the following growth and monotonicity conditions: for some $2 - 1/n < p \leqslant n$ there holds

$$\left|\mathcal{A}(x, \xi)\right| \leqslant \beta |\xi|^{p-1}, \tag{1.3}$$

$$\left\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \right\rangle \geqslant \alpha \left(|\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2 \tag{1.4}$$

for every $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$ and a.e. $x \in \mathbb{R}^n$. Here α and β are positive constants.

A typical example of such $\mathcal{A}$ is given by $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$ which gives rise to the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. However, in general no smoothness is assumed in the x-variable of the nonlinearity $\mathcal{A}$ throughout the paper.

Most of the results in this paper are obtained under a very mild condition on the domain Ω. That is the p-capacity uniform thickness condition (with constants $r_0, c_0 > 0$) imposed on $\mathbb{R}^n \setminus \Omega$. In this case we also say that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick with constants $r_0, c_0 > 0$. By definition this means that there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leqslant r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$ there holds

$$\operatorname{cap}_p\left(\overline{B_t(x)} \cap \left(\mathbb{R}^n \setminus \Omega\right), B_{2t}(x)\right) \geqslant c_0 \operatorname{cap}_p\left(\overline{B_t(x)}, B_{2t}(x)\right). \tag{1.5}$$

Here for a compact set $K \subset B_{2t}(x)$ we define its p-capacity by

$$\operatorname{cap}_p\left(K, B_{2t}(x)\right) = \inf\left\{ \int_{B_{2t}(x)} |\nabla\varphi|^p\, dy:\ \varphi \in C_0^\infty\left(B_{2t}(x)\right),\ \varphi \geqslant \chi_K \right\}.$$

It is easy to see that domains satisfying (1.5) include those with Lipschitz boundaries or even those that satisfy a uniform exterior corkscrew condition, where the latter means that there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leqslant r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$, there is $y \in B_t(x)$ such that $B_{t/c_0}(y) \subset \mathbb{R}^n \setminus \Omega$.

In this paper solutions u to the boundary value problems (1.1) and (1.2) are understood in the *renormalized sense*. It is well known that when the datum is not regular enough, a solution to nonlinear equations of Leray–Lions type does not necessarily belong to the natural Sobolev space $W_0^{1,p}(\Omega)$. This in particular brings up a major unsettling issue regarding the uniqueness of solutions. Therefore, we find it is most convenient to work with the notion of renormalized solutions (see [18,10,7]). However, for global estimates involving Eq. (1.1) it is enough to use a milder notion of solutions (see Remark 1.2 below). The notion of renormalized solutions will be recalled in the next section.

We now recall the definitions of Lorentz and Lorentz–Morrey spaces. For $0 < s < \infty$ and $0 < t \leqslant \infty$, the Lorentz space $L^{s,t}(\Omega)$ is the set of measurable functions g on Ω such that

$$\|g\|_{L^{s,t}(\Omega)} := \left[s \int_0^\infty \left(\alpha^s \left|\left\{x \in \Omega:\ \left|g(x)\right| > \alpha\right\}\right|\right)^{\frac{t}{s}} \frac{d\alpha}{\alpha} \right]^{\frac{1}{t}} < +\infty$$

when $t \neq \infty$; for $t = \infty$ the space $L^{s,\infty}(\Omega)$ is set to be the usual weak L^s or Marcinkiewicz space with quasinorm

$$\|g\|_{L^{s,\infty}(\Omega)} := \sup_{\alpha>0} \alpha \left|\left\{x \in \Omega:\ \left|g(x)\right| > \alpha\right\}\right|^{\frac{1}{s}}.$$

It is easy to see that when $t = s$ the Lorentz space $L^{s,s}(\Omega)$ is nothing but the Lebesgue space $L^s(\Omega)$. On the other hand, the Lorentz–Morrey function space $\mathcal{L}^{q,t;\theta}(\Omega)$, where $0 < \theta \leqslant n$, $0 < q < \infty$, $0 < t \leqslant \infty$, is the set of measurable functions g on Ω such that

$$\|g\|_{\mathcal{L}^{q,t;\theta}(\Omega)} := \sup_{\substack{0<r\leqslant \operatorname{diam}(\Omega) \\ z\in\Omega}} r^{\frac{\theta-n}{q}} \|g\|_{L^{q,t}(B_r(z)\cap\Omega)} < +\infty.$$

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