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Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems

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Abstract

Let $X \to Y$ be a fibration whose fibers are complete intersections of r quadrics. We develop new categorical and algebraic tools—a theory of relative homological projective duality and the Morita invariance of the even Clifford algebra under quadric reduction by hyperbolic splitting—to study semiorthogonal decompositions of the bounded derived category $\mathsf{D}^b(X)$. Together with results in the theory of quadratic forms, we apply these tools in the case where $r = 2$ and $X \to Y$ has relative dimension 1, 2, or 3, in which case the fibers are curves of genus one, Del Pezzo surfaces of degree 4, or Fano threefolds, respectively. In the latter two cases, if $Y = \mathbb{P}^1$ over an algebraically closed field of characteristic zero, we relate rationality questions to categorical representability of X.

Résumé

Soit $X \to Y$ une fibration en intersections complètes de r quadriques. Nous étudions une décomposition semiorthogonale de la catégorie dérivée $\mathsf{D}^b(X)$ grâce à des nouveaux outils catégoriques et algébriques : une théorie relative de la dualité homologique projective et l'invariance par équivalence de Morita de l'algèbre de Clifford paire sous un scindage hyperbolique. Avec des résultats sur les formes quadratiques, ceci nous permet d'étudier en détail les cas $r = 2$ et $X \to Y$ de dimension relative 1, 2, ou 3. Dans ces cas, les fibres sont respectivement des courbes de genre 1, des surfaces de Del Pezzo de degré 4, ou des variétés de Fano de dimension 3. Dans les deux derniers cas, si $Y = \mathbb{P}^1$ et si le corps de base est algébriquement clos de caractéristique zero, les résultats developpés nous permettent de mettre en relation la rationalité de X et sa représentabilité catégorique.

MSC: 14F05; 14E08; 11E08; 11E20; 11E88; 14F22; 14J26; 14M17; 15A66

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0. Introduction

One of the numerous applications of the study of triangulated categories in algebraic geometry is understanding how to extract, from the bounded derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(X)$, information about the birational geometry of a given smooth projective variety X.

Since the seminal work of Bondal and Orlov [21], it has become understood that such information should be encoded in semiorthogonal decompositions

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{A}_1, \dots, \mathsf{A}_n \rangle$$

by admissible triangulated subcategories: purely homological properties of the components of such a decomposition often reflect geometric properties of X. For example, if for each $i > 1$, the component A_i is "zero-dimensional" (i.e., equivalent to the bounded derived category of the base field), then A_1 should contain nontrivial information about the birational geometry of X. When X is a Fano threefold, many examples support this idea [12,13,21,59,60].

In particular, in the case that $X \to S$ is a conic bundle over a rational complex surface, a semiorthogonal decomposition by derived categories of points and smooth projective curves allows one to reconstruct the intermediate Jacobian $J(X)$ as the sum of the Jacobians of the curves. This can determine the rationality of X when S is minimal [13]. More generally, this works if X is a complex threefold with negative Kodaira dimension (e.g., a Fano threefold) whose codimension 2 cycles are universally described by a principally polarized abelian variety [14, Section 3.2]. In such cases, homological properties of semiorthogonal decompositions are related to classical notions of representability of cycles on X.

Attempting to trace the link between derived categories and algebraic cycles, the second and third named authors defined in [14] the notion of *categorical representability* in a given dimension m (or codimension $\dim(X) - m$) of a smooth projective variety X, by requiring the existence of a semiorthogonal decomposition whose components can be fully and faithfully embedded in derived categories of smooth projective varieties of dimension at most m. Categorical representability in dimension one is equivalent to the existence of a semiorthogonal decomposition by copies of the derived category of a point and derived categories of smooth projective curves. One might wonder if categorical representability in codimension 2 is a necessary condition for rationality. The work of Kuznetsov on cubic fourfolds [64], developed before the definition of categorical representability, shows how this philosophy persists as a tool to conjecturally understand rationality problems in dimension larger than three, where one cannot appeal to more classical methods, such as the study of the intermediate Jacobian.

In this paper, we provide two new instances where categorical representability is strictly related to birational properties. These arise as fibrations $X \to \mathbb{P}^1$ whose fibers are complete intersections of two quadrics. We impose a genericity hypothesis on such fibrations (see Definition 1.2.4) so that the associated pencil of quadrics has simple degeneration along a smooth divisor.

In Section 4, we consider fibrations $X \to \mathbb{P}^1$ whose fibers are Del Pezzo surfaces of degree four. Such threefolds have negative Kodaira dimension and their rationality (over the complex numbers) is completely classified [1,83]. We provide a purely categorical criterion for rationality of X based on [13].

Theorem 1. *(See Section 4.) Let $X \to \mathbb{P}^1$ be a generic Del Pezzo fibration of degree four over the complex numbers. Then X is rational if and only if it is categorically representable in codimension 2. Moreover, there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(X) = \big\langle \mathsf{D}^{\mathrm{b}}(\Gamma_1), \dots, \mathsf{D}^{\mathrm{b}}(\Gamma_k), E_1, \dots, E_l \big\rangle,$$

with Γ_i smooth projective curves and E_i exceptional objects if and only if $J(X) = \bigoplus J(\Gamma_i)$ as principally polarized abelian varieties.

In Section 5, we consider fibrations $X \to \mathbb{P}^1$ whose fibers are complete intersections of two four-dimensional quadrics. Such fourfolds have a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{A}_X, E_1, \dots, E_4 \rangle,$$

where E_i are exceptional objects. Moreover, we construct a fibration $T \to \mathbb{P}^1$ in hyperelliptic curves and a Brauer class $\beta \in \mathrm{Br}(T)$ such that $\mathsf{A}_X \simeq \mathsf{D}^{\mathrm{b}}(T, \beta)$. We state a conjecture in the same spirit as Kuznetsov's conjecture for cubic fourfolds [64, Conj. 1.1].

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