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Differential equations with singular fields

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Abstract

This paper investigates the well posedness of ordinary differential equations and more precisely the existence (or uniqueness) of a flow through explicit compactness estimates. Instead of assuming a bounded divergence condition on the vector field, a compressibility condition on the flow (bounded Jacobian) is considered. The main result provides existence under the condition that the vector field belongs to *BV* in dimension 2 and *SBV* in higher dimensions.

Résumé

Cet article étudie le caractère bien posé d'équations différentielles ordinaires et plus précisémment l'existence (ou l'unicité) d'un flot par des estimations directes de compacité. Une condition de compressibilité sur le flot est supposé au lieu d'une borne sur la divergence du champ de vitesse. Le principal résultat obtenu garantit l'existence sous l'hypothèse que le champ de vitesse est à variations bornées en dimension 2 ou dans l'espace *SBV* en dimensions supérieures.

MSC: 34C11; 35L45; 37C10

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1. Introduction

This article studies the existence (and secondary uniqueness) of a flow for the equation,

$$\partial_t X(t,x) = b\big(X(t,x)\big), \quad X(0,x) = x. \tag{1.1}$$

The most direct way to establish the existence of such of flow is of course through a simple approximation procedure. That means taking a regularized sequence $b_n \to b$, which enables to solve,

$$\partial_t X_n(t,x) = b_n\big(X(t,x)\big), \quad X_n(0,x) = x, \tag{1.2}$$

by the usual Cauchy–Lipschitz Theorem. To pass to the limit in (1.2) and obtain (1.1), it is enough to have compactness in some strong sense (in L^1_{loc} for instance) for the sequence X_n. Obviously some conditions are needed.

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First note that we are interested in flows, which means that we are looking for solutions X which are invertible: At least $JX = \det d_x X \neq 0$ a.e. (with $d_x X$ the differential of X in x only). So throughout this paper, only flows $x \to X(t,x)$ which are nearly incompressible are considered:

$$\frac{1}{C} \leqslant JX(t,x) \leqslant C, \quad \forall t \in [0,T],\ x \in \mathbb{R}^d. \tag{1.3}$$

If one obtains X as a limit of X_n then the most simple way of satisfying (1.3) is to have,

$$\frac{1}{C} \leqslant JX_n(t,x) \leqslant C, \quad \forall t \in [0,T],\ x \in \mathbb{R}^d, \tag{1.4}$$

for some constant C independent of n. Note that both conditions are only required on a finite and given time interval $[0,T]$ since one may easily extend X over $\mathbb{R}_+$ by the semi-group relation $X(t+T,x) = X(t, X(T,x))$. Usually (1.3) and (1.4) are obtained by assuming a bounded divergence condition on b or b_n but this is not the case here.

It is certainly difficult to guess what is the optimal condition on b. It is currently thought that $b \in BV(\mathbb{R}^d)$ is enough or (see [12])

Bressan's compactness conjecture. *Let X_n be regular* (C^1) *solutions to* (1.2)*, satisfying* (1.4) *and with* $\sup_n \int_{\mathbb{R}^d} |db_n(x)|\,dx < \infty$. *Then the sequence X_n is locally compact in* $L^1([0,T]\times\mathbb{R}^d)$.

From this, one would directly obtain the existence of a flow to (1.1) provided that $b \in BV(\mathbb{R}^d)$ and (1.3) holds. Instead of the full Bressan's conjecture, this article essentially recovers, through a different method, the result of [4] namely under the condition $b \in SBV$.

Theorem 1.1. *Assume that $b \in SBV_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with a locally finite jump set* (*for the* $(d-1)$-*dimensional Hausdorff measure*). *Let X_n be regular solutions to* (1.2)*, satisfying* (1.4) *and such that $b_n \to b$ belongs uniformly to $L^\infty(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$. Then X_n is locally compact in $L^1([0,T]\times\mathbb{R}^d)$.*

It is possible to be more precise only in dimension 2:

Theorem 1.2. *Assume that $d \leqslant 2$, $b \in BV_{\mathrm{loc}}(\mathbb{R}^d)$. Let X_n be regular solutions to* (1.2)*, satisfying* (1.4) *and such that $b_n \to b$ belongs uniformly to $L^\infty(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$ with $\inf_K b_n \cdot B > 0$ for any compact K and some $B \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Then X_n is locally compact in $L^1([0,T]\times\mathbb{R}^d)$.*

The proof of the first result is found in Section 7 and the proof of the second in Section 6. After notations and examples in Section 2 and technical lemmas in Section 3, particular cases are studied. In Section 4, a very simple proof is given if $b \in W^{1,1}$. Section 5 studies (1.1) in dimension 1 for which compactness holds under very general conditions (essentially nothing for b and a much weaker version of (1.3)). The final section offers some comments on the unresolved issues in the full BV case.

The question of uniqueness is deeply connected to the existence and in fact the proof of Theorem 1.1 may be slightly altered in order to provide it (it is more complicated for Theorem 1.2). Proofs are always given for the compactness of the sequence but it is indicated and briefly explained after the stated results whether they can also give uniqueness; This is usually the case except for Sections 5 and 6.

The well posedness of (1.1) is classically obtained by the Cauchy–Lipschitz Theorem. This is based on the simple estimate:

$$\bigl|X(t,x+\delta) - X(t,x)\bigr| \leqslant |\delta| e^{t\|db\|_{L^\infty}}. \tag{1.5}$$

Notice that a similar bound holds if b is only log-Lipschitz, leading to the important result of uniqueness for the $2d$ incompressible Euler system (see for instance [28] among many other references).

The idea in this article is to get (1.5) for *almost all* x. It is therefore greatly inspired by the recent approach developed in [18] (see also [17]), where the authors control the functional:

$$\int_{\mathbb{R}^d} \sup_r \int_{S^{d-1}} \log\left(1 + \frac{|X(t,x+rw) - X(t,x)|}{r}\right) dw\,dx. \tag{1.6}$$