



ELSEVIER

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



Mathematical analysis

On the maximal mean curvature of a smooth surface

*Sur la courbure moyenne maximale d'une surface lisse*

Vincenzo Ferone, Carlo Nitsch, Cristina Trombetti

Università degli Studi di Napoli Federico II, Napoli, Italy

ARTICLE INFO

Article history:

Received 20 April 2016

Accepted 26 May 2016

Available online 22 July 2016

Presented by Haïm Brézis

ABSTRACT

Given a smooth simply connected planar domain, the area is bounded away from zero in terms of the maximal curvature alone. We show that in higher dimensions this is not true, and for a given maximal mean curvature we provide smooth embeddings of the ball with arbitrary small volume.

© 2016 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Étant donné un domaine planaire simplement connexe lisse, l'aire est bornée loin de zéro en termes de la seule courbure maximale. Nous montrons que pour des dimensions plus élevées ce n'est pas vrai, et nous fournissons, pour un maximum donné de la courbure moyenne, des plongements lisses de la boule avec un petit volume arbitraire.

© 2016 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

According to a classical result [3,14], for any smooth simple planar curve γ , if the curvature κ is bounded from above by some positive constant M , then the curve γ encloses a bounded simply connected domain Ω that contains a disk of radius M^{-1} . In particular, see [12], if Ω^* is a disk having same measure as Ω , the following inequality holds true

$$\|\kappa\|_{L^\infty(\partial\Omega)} \geq \|\kappa\|_{L^\infty(\partial\Omega^*)} \quad (1)$$

or equivalently

$$\|\kappa\|_{L^\infty(\partial\Omega)}^2 \text{Area}(\Omega) \geq \pi,$$

equality holding in both cases if and only if Ω is a disk.

Very recently, such inequalities have been generalized to other L^p norm of the curvature. More precisely, for $p = 2$ (see [2,4,8]), and for $p \geq 1$ (see [5]) it holds

$$\|\kappa\|_{L^p(\partial\Omega)} \geq \|\kappa\|_{L^p(\partial\Omega^*)}$$

E-mail addresses: ferone@unina.it (V. Ferone), c.nitsch@unina.it (C. Nitsch), cristina@unina.it (C. Trombetti).

or equivalently

$$\|\kappa\|_{L^p(\partial\Omega)}^2 \text{Area}(\Omega)^{\frac{p-1}{p}} \geq 2^{2/p} \pi^{(p+1)/p},$$

equality holding again in both cases if and only if Ω is a disk.

In this short note, we consider the 3-dimensional case. We replace the simply connected planar domain by sets diffeomorphic to balls in \mathbb{R}^3 , and the planar curvature κ by the mean curvature H of the boundary. If by $S(\cdot)$ and $V(\cdot)$ we denote the surface area and the volume, respectively, our main result then reads as follows.

Theorem 1.1. *For any $\epsilon > 0$ there exists $\Omega_\epsilon \subset \mathbb{R}^3$, diffeomorphic to the unit ball, with smooth $C^{1,1}$ boundary, such that*

$$\|H\|_{L^\infty(\partial\Omega_\epsilon)} \leq 1, \quad |S(\Omega_\epsilon) - 8\pi| \leq \epsilon, \quad \text{and} \quad V(\Omega_\epsilon) \leq \epsilon.$$

If Ω^* denotes the ball having the same volume as Ω , a first consequence is as follows.

Corollary 1.2. *In the class of $C^{1,1}$ subsets of \mathbb{R}^3 diffeomorphic to balls for all $2 < p \leq \infty$, we have*

$$\inf \{ \|H\|_{L^p(\partial\Omega)} : V(\Omega) = 1 \} = 0. \tag{2}$$

In particular, for all $2 < p \leq \infty$, there exists a $C^{1,1}$ set $\Omega \subset \mathbb{R}^3$, diffeomorphic to the ball Ω^ , such that*

$$\|H\|_{L^p(\partial\Omega)} < \|H\|_{L^p(\partial\Omega^*)}.$$

Notice that $\|H\|_{L^2}$ is a very special case since it corresponds to the Willmore energy (invariant under dilation), which is indeed minimal on balls [16]. The case $p = 2$ is also a threshold case since $\|H\|_{L^p}$, for $p < 2$, scales under dilation as a positive power of the volume. Under volume constraint, $\|H\|_{L^p}$ is in fact bounded away from zero for all $1 \leq p < 2$, even if the optimal lower bounds are still unknown, see [9,15].

Our interest in this kind of inequalities is also due to a question arisen in [11,13] in relation to estimates for Laplacian eigenvalue with Robin boundary conditions.

For any given $\alpha > 0$, consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega. \end{cases} \tag{3}$$

By $\lambda(\Omega, \alpha)$ we denote the greatest (negative) λ such that (3) admits a nontrivial solution, namely:

$$\lambda(\Omega, \alpha) = \max \left\{ \alpha \int_{\partial\Omega} v^2 - \int_{\Omega} |\nabla v|^2 : v \in H^1(\Omega), \int_{\Omega} v^2 = 1 \right\}.$$

It has been conjectured for long time [1] that balls achieve the greatest eigenvalue among sets of given measure. Indeed they are local maximizers in any dimension (see [6]) and global maximizers in 2 dimensions for α small enough (see [7]). However, in [7] the authors also show that large values of α provides the annulus as a counterexample to the conjecture.

In [11,13], this was clarified showing that whenever $\Omega \subset \mathbb{R}^n$ is $C^{1,1}$ then

$$\lambda(\Omega, \alpha) = -\alpha^2 - \alpha(n-1) \sup_{\partial\Omega} H + o(\alpha) \quad \text{as } \alpha \rightarrow \infty.$$

In fact, since any annulus \mathcal{A} having same measure as a ball B also has a smaller maximal curvature, we have $\lambda(\mathcal{A}, \alpha) > \lambda(B, \alpha)$ as soon as α is large enough. Thereafter, in [13] the authors were interested in minimizing the maximal curvature in classes of domains of given volume subject to some kind of additional topological constraints. In view of (1), in two-dimension balls achieve the minimal maximal mean curvature whenever we restrict to simply connected sets. In dimensions greater than 2, they were able as well to prove that starshapedness is enough to get the same result. Whence they left open the following problem.

Question. *Let $\Omega \in \mathbb{R}^n$ be a bounded smooth domain with a connected boundary and let Ω^* be a ball with same volume. Do we have $\sup_{\partial\Omega} H \geq \sup_{\partial\Omega^*} H$?*

Corollary 1.2 provides a negative answer.

Download English Version:

<https://daneshyari.com/en/article/10180903>

Download Persian Version:

<https://daneshyari.com/article/10180903>

[Daneshyari.com](https://daneshyari.com)