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Algebraic geometry

The boundary of the orbit of the 3-by-3 determinant polynomial ☆

La frontière de l'orbite du polynôme déterminant 3 par 3

Jesko Hüttenhain, Pierre Lairez

Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

A R T I C L E   I N F O

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A B S T R A C T

We consider the 3×3 determinant polynomial and we describe the limit points of the set of all polynomials obtained from the determinant polynomial by linear change of variables. This answers a question of Joseph M. Landsberg.

R É S U M É

Nous étudions le polynôme donné par le déterminant 3×3 et décrivons l'adhérence de l'ensemble des polynômes obtenus par changements de variables linéaires à partir de ce déterminant, ce qui répond à une question de Joseph M. Lansberg.

0. Introduction

Mulmuley and Sohoni [11] propose, in their *geometric complexity theory*, to study the geometry of the orbit closure of some polynomials under linear change of variables, and especially, the determinant polynomial. Yet, very few explicit results describing the geometry are known in low dimension. The purpose of this work is to describe the boundary of the orbit of the 3×3 determinant, that is, the set of limit points of the orbit that are not in the orbit.

Let $\det_3$ be the polynomial

$$\det_3 \overset{\text{def}}{=} \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \in \mathbb{C}[x_1, \dots, x_9],$$

which we consider as a homogeneous form of degree 3 on the space $\mathbb{C}^{3\times 3}$ of 3×3 matrices, denoted W. Let $\mathbb{C}[W]_3$ denote the 165-dimensional space of all homogeneous forms of degree 3 on W. The group $\mathrm{GL}(W)$ acts on $\mathbb{C}[W]_3$ by right

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E-mail addresses: jesko@math.tu-berlin.de (J. Hüttenhain), pierre@lairez.fr (P. Lairez).

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composition. For a nonzero $P \in \mathbb{C}[W]_3$, let $\Omega(P)$ denote the (projective) orbit of P, namely the set of all $[P \circ a] \in \mathbb{P}(\mathbb{C}[W]_3)$, with $a \in \mathrm{GL}(W)$. The *boundary* of the orbit of P, denoted $\partial\Omega(P)$, is $\overline{\Omega(P)} \setminus \Omega(P)$, where $\overline{\Omega(P)}$, denoted also $\overline{\Omega}(P)$, is the Zariski closure of the orbit in $\mathbb{P}(\mathbb{C}[W]_3)$.

Our main result is a description of $\partial\Omega(\det_3)$ that answers a question of Landsberg [10, Problem 5.4]: the two known components are the only ones. In §1, we explain the construction of the two components. Our contribution lies in §2 where we show that there is no other component.

Theorem 1. *The boundary $\partial\Omega(\det_3)$ has exactly two irreducible components:*

- *The orbit closure of the determinant of the generic traceless matrix, namely*

$$P_1 \stackrel{\text{def}}{=} \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & -x_1 - x_5 \end{pmatrix};$$

- *The orbit closure of the universal homogeneous polynomial of degree two in three variables, namely*

$$P_2 \stackrel{\text{def}}{=} x_4 \cdot x_1^2 + x_5 \cdot x_2^2 + x_6 \cdot x_3^2 + x_7 \cdot x_1 x_2 + x_8 \cdot x_2 x_3 + x_9 \cdot x_1 x_3.$$

The two components are different in nature: the first one is the orbit closure of a polynomial in only eight variables and is included in the orbit of $[\det_3]$ under the action of $\operatorname{End} W$; the second is more subtle and is not included in the $\operatorname{End} W$-orbit of $[\det_3]$. Both components have analogues in higher dimension and some results are known about them [9].

1. Construction of two components of the boundary

For $P \in \mathbb{C}[W]_3 \setminus \{0\}$, let $H(P) \subset \mathrm{GL}(W)$ denote its stabilizer, that is

$$H(P) \stackrel{\text{def}}{=} \{a \in \mathrm{GL}(W) \mid P \circ a = P\}.$$

The stabilizer $H(\det_3)$ is generated by the transposition map $A \mapsto A^{\mathrm{T}}$ and the maps $A \mapsto UAV$, with U and V in $\mathrm{SL}(\mathbb{C}^3)$ [3].

Lemma 2. *For any $P \in \mathbb{C}[W]_3$, $\dim \Omega(P) = 80 - \dim H(P)$. In particular, $\dim \Omega(\det_3) = 64$ and $\dim \Omega(P_1) = \dim \Omega(P_2) = 63$.*

Proof. An easy application of the fiber dimension theorem to the map $a \in \mathrm{GL}(W) \mapsto P \circ a \in \mathbb{C}[W]_3$ gives that the dimension of the orbit of P in $\mathbb{C}[W]_3$ is $81 - \dim H(P)$. Since the projective orbit in $\mathbb{P}(\mathbb{C}[W]_3)$ has one dimension less, the first claim follows.

The stabilizer $H(\det_3)$ has dimension 16, hence $\dim \Omega(\det_3) = 64$. To compute the dimension of $H(P_i)$, $1 \leqslant i \leqslant 2$, one can compute the dimension of its Lie algebra defined as

$$T_1 H(P_i) = \left\{ a \in \operatorname{End}(W) \;\middle|\; P(x + ta(x)) = P(x) + O(t^2) \right\}.$$

It amounts to computing the nullspace of a 165×81 matrix, which is easy using a computer. □

Lemma 3. *The boundary $\partial\Omega(\det_3)$ is pure of dimension* 63.

Proof. Let $\Omega'(\det_3)$ be the affine orbit of $\det_3$ in $\mathbb{C}[W]_3$ under the action of $\mathrm{GL}(W)$. It is isomorphic to $\mathrm{GL}(W)/H(\det_3)$, which is an affine variety because $H(\det_3)$ is reductive [12, §4.2]. Therefore $\Omega'(\det_3)$ is an affine open subset of its closure, it follows that the complement of $\Omega'(\det_3)$ in its closure is pure of codimension 1 [7, Corollaire 21.12.7], and the same holds true after projectivization. □

Let φ be the rational map

$$\varphi : [a] \in \mathbb{P}(\operatorname{End} W) \dashrightarrow [\det_3 \circ a] \in \overline{\Omega}(\det_3). \tag{1}$$

Let also Z be the irreducible hypersurface of $\mathbb{P}(\operatorname{End} W)$

$$Z \stackrel{\text{def}}{=} \{[a] \in \mathbb{P}(\operatorname{End} W) \mid \det(a) = 0\}.$$

Note the difference between $\det_3 \circ a$, which is a regular function of W, and $\det(a)$, which is a scalar. The indeterminacy locus of φ is a strict subset of Z. By definition, $\Omega(\det_3) = \varphi(\mathbb{P}(\operatorname{End} W) \setminus Z)$. Let $\varphi(Z)$ denote the image of the set of the points of Z where φ is defined.

Lemma 4. *The closure $\overline{\varphi(Z)}$ is an irreducible component of $\partial\Omega(\det_3)$. Furthermore $\overline{\varphi(Z)} = \overline{\Omega}(P_1)$.*