C. R. Acad. Sci. Paris, Ser. I 354 (2016) 419–423

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential geometry/Dynamical systems

Remarks on the symplectic invariance of Aubry–Mather sets

Remarques sur l'invariance symplectique des ensembles d'Aubry–Mather

Marco Mazzucchelli [a], Alfonso Sorrentino [b]

[a] *CNRS and UMPA, École normale Supérieure de Lyon, France*
[b] *Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Italy*

A R T I C L E   I N F O

Article history:
Received 6 May 2015
Accepted 12 January 2016
Available online 11 February 2016

Presented by Charles-Michel Marle

A B S T R A C T

In this note, we discuss and clarify some issues related to the generalization of Bernard's theorem on the symplectic invariance of Aubry, Mather and Mañé sets, to the cases of non-zero cohomology classes or non-exact symplectomorphisms, not necessarily homotopic to the identity.

R É S U M É

On discute et clarifie quelques questions liées à la généralisation du théorème de Bernard sur l'invariance symplectique des ensembles d'Aubry, de Mather et de Mañé aux cas de classes de cohomologie non nulles et de symplectomorphismes non exacts et non nécessairement homotopes à l'identité.

1. Introduction

In the study of Hamiltonian dynamical systems, *Aubry–Mather theory* refers to a series of variational techniques, related to the Principle of Least Action, that singled out particular orbits and, more generally, invariant sets obtained as minimizing solutions to a variational problem. These sets are nowadays called the *Mather*, *Aubry* and *Mañé sets*, and as a result of their action-minimizing property, they enjoy many interesting dynamical properties and a rich geometric structure.

Symplectic aspects of Aubry–Mather theory and its relation to symplectic geometry have soon attracted a lot of interest, starting from the work of Paternain, Polterovich, and Siburg [7]. In his seminal paper [1], Bernard established the symplectic invariance of the Aubry–Mather sets corresponding to the zero cohomology class, under the action of exact symplectomorphisms that preserve the condition of being of Tonelli type. See also, just to mention a few papers in the literature that followed [2,3,8–12].

In this note, starting from Bernard's result, we would like to discuss and clarify some aspects related to the generalization of his theorem to other cohomology classes and to non-exact symplectomorphisms, not necessarily homotopic to the

E-mail addresses: marco.mazzucchelli@ens-lyon.fr (M. Mazzucchelli), sorrentino@mat.uniroma2.it (A. Sorrentino).

http://dx.doi.org/10.1016/j.crma.2016.01.001
1631-073X/

identity (see Theorem 9 and Corollary 10). As we shall see, in fact, one has to keep into account two distinct issues: the cohomology class of the symplectomorphism, as well as the action of the symplectomorphism on de Rham cohomology classes.

2. Notation and setting

Let M be a closed manifold, and let us denote by $\mathrm{T}M$ and T^*M, respectively, its tangent and cotangent bundles. A *Tonelli Hamiltonian* is a C^2 function $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ that is strictly convex and superlinear in each fiber. For each de Rham cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, we denote by $\mathcal{M}^*_c(H)$, $\mathcal{A}^*_c(H)$ and $\mathcal{N}^*_c(H)$, respectively the *Mather*, *Aubry* and *Mañé* sets of cohomology class c, associated with H. Moreover, we denote by $\alpha_H : \mathrm{H}^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ and $\beta_H : \mathrm{H}_1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ the so-called *Mather's minimal average actions*. We refer to [5,6,9] for a precise definition of these objects and for a discussion of their properties.

Let λ be the Liouville form on T^*M, which can be written in local coordinates as $\sum_j p_j\,\mathrm{d}q_j$. The projection onto the base $\pi : \mathrm{T}M \longrightarrow M$ is a homotopy equivalence whose homotopy inverse is given by the inclusion of the 0-section $\iota : M \hookrightarrow \mathrm{T}^*M$. From now on, we tacitly identify the de Rham cohomology groups $\mathrm{H}^1(M;\mathbb{R})$ and $\mathrm{H}^1(\mathrm{T}^*M;\mathbb{R})$ by means of the isomorphisms induced by π and ι. Analogously, we identify the singular homology groups $\mathrm{H}_1(M;\mathbb{R})$ and $\mathrm{H}_1(\mathrm{T}^*M;\mathbb{R})$.

Given a symplectomorphism Ψ of $(\mathrm{T}^*M, \mathrm{d}\lambda)$, we denote by $[[\Psi]]$ the cohomology class $[\Psi^*\lambda - \lambda] \in \mathrm{H}^1(M;\mathbb{R})$. Such a symplectomorphism is called *exact* when $[[\Psi]] = 0$.

Example 1.

(i) A particularly simple class of symplectomorphisms is given by translations in the fibers, that is, maps $\Theta_\alpha(q,p) = (q, p + \alpha_q)$, where α is a closed 1-form on M. These symplectomorphisms are obviously homotopic to the identity, and their cohomology class is given by $[[\Theta_\alpha]] = [\alpha]$.

(ii) Any diffeomorphism $\psi : M \longrightarrow M$ can be lifted to a diffeomorphism $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ defined by

$$\Psi(q,p) := (\psi(q), (\psi^{-1})^* p) = \Big(\psi(q), p \circ \mathrm{d}\psi^{-1}(\psi(q))\Big),$$

that preserves the Liouville form λ, *i.e.*, $\Psi^*\lambda = \lambda$. In particular, Ψ is an exact symplectomorphism. As a special instance, let $A \in \mathrm{GL}_n(\mathbb{Z})$ and consider the linear map on $\mathbb{T}^n$ given by $\psi(q) = (A^{\mathrm{T}})^{-1} q$. The associated symplectomorphism of $\mathrm{T}^*\mathbb{T}^n$ is given by $\Psi(q,p) = ((A^{\mathrm{T}})^{-1} q, Ap)$.

3. Symplectic aspects of Aubry–Mather theory

Let us start by recalling Bernard's result.

Theorem 2. *(Bernard [1].) Let $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ be a Tonelli Hamiltonian and $\Phi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ an exact symplectomorphism such that $H \circ \Phi$ is still of Tonelli type. Then:*

$$\begin{aligned}
\mathcal{M}^*_0(H \circ \Phi) &= \Phi^{-1}\big(\mathcal{M}^*_0(H)\big)\\
\mathcal{A}^*_0(H \circ \Phi) &= \Phi^{-1}\big(\mathcal{A}^*_0(H)\big)\\
\mathcal{N}^*_0(H \circ \Phi) &= \Phi^{-1}\big(\mathcal{N}^*_0(H)\big).
\end{aligned}$$

Remark 3. Obviously the condition that the Hamiltonian $H \circ \Phi$ be still of Tonelli type is very restrictive. For instance, if $M = S^1$ and $H(q,p) = p^2$, consider any Hamiltonian diffeomorphism $\Phi : \mathrm{T}^*S^1 \longrightarrow \mathrm{T}^*S^1$ mapping a fiber $\mathrm{T}^*_q S^1$ to a curve $(q(t), p(t))$ such that $t \mapsto p(t)$ is not monotone; then the composition $H \circ \Phi$ is not Tonelli, as its restriction to the fiber $\mathrm{T}^*_q S^1$ is not convex.

Remark 4.

(i) Bernard's theorem does not hold anymore if Φ is not exact. For example, if we consider the symplectomorphism $\Theta_\alpha(q,p) = (q, p + \alpha_q)$, where α is a closed 1-form on M, then one can easily check that

$$\mathcal{M}^*_0(H \circ \Theta_\alpha) = \Theta_\alpha^{-1}\big(\mathcal{M}^*_{[\alpha]}(H)\big),$$

which, in general, could be different from $\Theta_\alpha^{-1}\big(\mathcal{M}^*_0(H)\big)$. Similarly, for the Aubry and Mañé sets (see Proposition 7).

(ii) Even in the case of exact symplectomorphisms, Bernard's theorem may fail for non-zero cohomology classes. For example, let us consider a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ and consider the exact symplectomorphism

$$\Psi(q,p) = ((A^{\mathrm{T}})^{-1} q, Ap),$$