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Partial differential equations/Calculus of variations

Higher-order topological sensitivity analysis for the Laplace operator

Analyse de sensibilité topologique d'ordre supérieur pour l'opérateur de Laplace

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A R T I C L E   I N F O

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A B S T R A C T

This paper deals with higher-order topological sensitivity analysis for the Laplace operator with respect to the presence of a Dirichlet geometry perturbation. Two main results are presented in this work. In the first one, we discuss the influence of the considered geometry perturbation on the Laplace solution. The second one is devoted to the higher-order topological derivatives. We derive a higher-order topological sensitivity analysis for a large class of shape functions.

R É S U M É

Dans ce papier, on donne une analyse de sensibilité pour l'opérateur de Laplace par rapport à des perturbations géométriques de type Dirichlet. On pésente deux résultats. Le premier concerne l'influence de la perturbation géométrique sur la solution du problème de Laplace. On dérive une formule de représentation asymptotique d'ordre supérieur décrivant le comportement de la solution perturbée en fonction de la taille de la perturbation. Le deuxième concerne les dérivées d'une fonction de forme par rapport à la modification de la topologie du domaine. On donne un développement asymtotique topologique d'ordre supérieur valable pour une grande classe de fonctions de forme.

1. Introduction

Topological sensitivity analysis has been derived for various operators and has been used for many topology optimization problems, e.g. for the Laplace equation [9], for the Stokes system [1,2,4,12], for the elasticity problem [8,11], for the

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Helmholtz equation [15], for the elastodynamic and acoustic problems [5,6,10], etc. In all these works, the optimization algorithms are based on the first-order topological derivative, which is only valid for small geometry perturbation size. The use of higher-order terms in the topological asymptotic expansion of the shape function may certainly be decisive in improving the topological optimization algorithms without restrictions on the perturbation sizes. This question has been partially addressed by Novotny et al. [13,14] in the particular case of circular holes with an asymptotic expansion limited to order two. The proposed mathematical analysis in [13,14] is based on a restricted approach and cannot be generalized to the higher-order case.

In this work, we consider the three-dimensional case and we derive a higher-order topological sensitivity analysis for the Laplace operator with respect to the presence of Dirichlet geometric perturbations. More precisely, let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We consider the case in which Ω contains a geometry perturbation $\omega_{z,\varepsilon}$ that is centered at $z \in \Omega$ and has the form $\omega_{z,\varepsilon} = z + \varepsilon\omega$, where $\omega \subset \mathbb{R}^3$ is a given fixed and bounded regular domain containing the origin.

Two main questions are discussed in this paper. The first one concerns the influence of the geometry perturbation on the Laplace equation solution. We derive a higher-order asymptotic expansion for the solution to the perturbed Laplace equation with respect to the geometry perturbation size. This question has been investigated by Ammari and Kang [3] in the inhomogeneity case where the perturbed solution is computed in the entire domain Ω using a continuity condition on the boundary $\partial\omega_{z,\varepsilon}$. In this work, we deal with more singular geometric perturbations. The solution to the perturbed Laplace equation is computed in $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\omega_{z,\varepsilon}}$ with Dirichlet condition on $\partial\omega_{z,\varepsilon}$. As we will show in Section 3, this type of perturbations leads to an asymptotic behavior with respect to ε different from that obtained in [3].

The second one concerns the higher-order topological derivatives. In Section 4, we derive a higher-order topological asymptotic expansion for the Laplace operator. More precisely, we derive an asymptotic expansion of a given shape functional j in the following form:

$$j(\Omega_{z,\varepsilon}) = j(\Omega) + \sum_{k=1}^{N} f_k(\varepsilon)\delta^k j(z) + o(f_N(\varepsilon)), \text{ where}$$

- $(f_k)_{1\le k\le N}$ are positive scalar functions verifying $f_{k+1}(\varepsilon) = o(f_k(\varepsilon))$ and $\lim_{\varepsilon\to 0} f_k(\varepsilon) = 0$.
- $\delta^k j$ denotes the kth topological derivative of the shape function j.

2. Formulation of the problem

Consider a shape function j of the form

$$j(\Omega \setminus \overline{\omega_{z,\varepsilon}}) = J_\varepsilon(u_\varepsilon),$$

where J_ε is defined on $H^1(\Omega \setminus \overline{\omega_{z,\varepsilon}})$ and u_ε is the solution to Laplace problem in the perturbed domain $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\omega_{z,\varepsilon}}$ with homogeneous Dirichlet condition on $\partial\omega_{z,\varepsilon}$

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega_{z,\varepsilon}, \\ \nabla u_\varepsilon \cdot n &= \Phi_n && \text{on } \Gamma_n, \\ u_\varepsilon &= \Phi_d && \text{on } \Gamma_d, \\ u_\varepsilon &= 0 && \text{on } \partial\omega_{z,\varepsilon}, \end{aligned} \tag{1}$$

where $\Phi_n \in H^{-1/2}(\Gamma_n)$ and $\Phi_d \in H^{1/2}(\Gamma_d)$ are two given data, with Γ_n and Γ_d are two parts of the boundary $\partial\Omega$ verifying $\overline{\partial\Omega} = \overline{\Gamma_n} \cup \overline{\Gamma_d}$ and $\Gamma_d \cap \Gamma_n = \emptyset$.

As we have mentioned in the introduction, the aim of this work is to derive a higher-order topological asymptotic expansion for the shape function j with respect to the presence of the geometry perturbation $\omega_{z,\varepsilon}$ in the domain Ω.

3. Sensitivity analysis for the Laplace operator

In this section, we give a sensitivity analysis for the Laplace solution with respect to the presence of a geometry perturbation $\omega_{z,\varepsilon}$ in the domain Ω. More precisely, we derive an asymptotic expansion for the solution u_ε with respect to the perturbation size ε. We start our analysis by the following estimate.

Lemma 3.1. *If the geometry perturbation $\omega_{z,\varepsilon} \subset \Omega$ is not close to the boundary $\partial\Omega$, then the variation $u_\varepsilon - u_0$ admits the following estimate:*

$$u_\varepsilon(x) - u_0(x) = W_0((x-z)/\varepsilon) + O(\varepsilon) \text{ in } \Omega_{z,\varepsilon},$$

where the function $x \longmapsto W_0((x-z)/\varepsilon)$ is the unique solution to the Laplace exterior problem