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Partial differential equations

Note to the problem of the asymptotic behavior of a viscous incompressible flow around a rotating body

Remarque sur le problème du comportement asymptotique de l'écoulement d'un fluide incompressible autour d'un corps rigide en rotation

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ABSTRACT

We consider weak solutions to the stationary Navier–Stokes system with Oseen and rotational terms, in an exterior domain. We are interested in the leading term for the velocity field and its gradient. Moreover, we deal with the asymptotic behavior at infinity. We proved that the velocity may be split, within constants, into the first column of the fundamental solution to the Oseen system, plus a remainder term decaying pointwise near infinity at a rate which is higher than the decay rate of the Oseen tensor. This result improves the theory proposed by M. Kyed [Asymptotic profile of a linearized flow past a rotating body, Q. Appl. Math. 71 (2013) 489–500; On the asymptotic structure of a Navier–Stokes flow past a rotating body, J. Math. Soc. Jpn. 66 (2014) 1–16].

RÉSUMÉ

Nous considérons des solutions faibles du système de Navier–Stokes stationnaire avec un terme d'Oseen et des termes rotationnels dans un domaine extérieur. Notre intérêt se porte sur la partie principale d'un développement asymptotique de la vitesse et de son gradient. Nous montrons que la vitesse peut être scindée, à des constantes près, en la première colonne de la solution fondamentale du système d'Oseen (« tenseur d'Oseen »), plus un reste qui décroît ponctuellement dans un voisinage d'infini, à un taux qui est plus élevé que le taux de décroissance du tenseur d'Oseen. Ce résultat améliore la théorie présentée par M. Kyed [Asymptotic profile of a linearized flow past a rotating body, Q. Appl. Math. 71 (2013) 489–500 ; On the asymptotic structure of a Navier–Stokes flow past a rotating body, J. Math. Soc. Jpn. 66 (2014) 1–16].

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1. Introduction

We shall consider the following problem. Let $\mathfrak{D} \subset \mathbb{R}^3$ be an open bounded set. Suppose this set describes a rigid body moving with constant nonzero translational and angular velocity in an incompressible viscous fluid. Then the flow around this body with respect to a frame attached to this body is governed by the following set of non-dimensional equations (see [7]):

$$-\Delta u + \tau\,\partial_1 u + \tau\,(u\cdot\nabla)u - (\omega \times x)\cdot\nabla u + \omega\times u + \nabla\pi = f,\;\; \operatorname{div} u = 0, \tag{1}$$

in the exterior domain $\overline{\mathfrak{D}}^c := \mathbb{R}^3\backslash\overline{\mathfrak{D}}$, supplemented by a decay condition at infinity,

$$u(x) \to 0 \;\text{ for }\; |x| \to \infty, \tag{2}$$

and suitable boundary conditions on $\partial\mathfrak{D}$.

In (1) and (2), the functions $u : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}^3$ and $\pi : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}$ are the unknown relative velocity and pressure field of the fluid, respectively, whereas the function $f : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}^3$ stands for a prescribed volume force acting on the fluid. The vector $\tau\,(-1,0,0)$ represents the uniform velocity of the flow at infinity or the velocity of the body, depending on the physical situation, and $\omega := \varrho\cdot(1,0,0)$ corresponds to the constant angular velocity of the body. In particular, the translational and angular velocity vectors are parallel. From a physical point of view, this assumption is natural for a steady flow. The parameters $\tau \in (0,\infty)$ and $\varrho \in \mathbb{R}\backslash\{0\}$ are dimensionless quantities that can be identified with the Reynolds and Taylor numbers, respectively. They will be considered as fixed, like the domain $\mathfrak{D}$.

We are interested in “Leray solutions” to (1), (2), that is weak solutions characterized by the conditions $u \in L^6(\overline{\mathfrak{D}}^c)^3 \cap W^{1,1}_{\text{loc}}(\overline{\mathfrak{D}}^c)^3$, $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$ and $\pi \in L^2_{\text{loc}}(\overline{\mathfrak{D}}^c)$.

From [8] and [3], it follows that the velocity part u of a Leray solution (u,π) to (1), (2) decays for $|x| \to \infty$ as expressed by the estimates

$$|u(x)| \le C\,\big(|x|\,s(x)\big)^{-1}, \quad |\nabla u(x)| \le C\,\big(|x|\,s(x)\big)^{-3/2} \tag{3}$$

for $x \in \mathbb{R}^3$ with $|x|$ sufficiently large, where $s(x) := 1 + |x| - x_1$ $(x \in \mathbb{R}^3)$ and $C > 0$ a constant independent of x. The factor $s(x)$ may be considered as a mathematical manifestation of the wake extending downstream behind a body moving in a viscous fluid.

By Kyed [10], it was shown that

$$u_j(x) = \gamma\,E_{j1}(x) + R_j(x), \quad \partial_l u_j(x) = \gamma\,\partial_l E_{j1}(x) + S_{jl}(x)\;\; (x \in \overline{\mathfrak{D}}^c,\; 1 \le j,l \le 3), \tag{4}$$

where $E : \mathbb{R}^3\backslash\{0\} \mapsto \mathbb{R}^4 \times \mathbb{R}^3$ denotes a fundamental solution to the Oseen system

$$-\Delta v + \tau\,\partial_1 v + \nabla\Pi = f,\;\; \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3. \tag{5}$$

The definition of the function E is stated in Section 2. As becomes apparent from this definition, the term $E_{j1}(x)$ may be expressed explicitly in terms of elementary functions. The coefficient γ is also given explicitly, its definition involving the Cauchy stress tensor. The remainder terms R and S are characterized by the relations $R \in L^q(\overline{\mathfrak{D}}^c)^3$ for $q \in (4/3,\infty)$, $S \in L^q(\overline{\mathfrak{D}}^c)^3$ for $q \in (1,\infty)$. Since it is known from [6, Section VII.3] that $E_{j1}|B_r^c \notin L^q(B_r^c)$ for $r > 0$, $q \in [1,2]$, and $\partial_l E_{j1}|B_r^c \notin L^q(B_r^c)$ for $r > 0$, $q \in [1, 4/3]$, $j,l \in \{1, 2, 3\}$, the function R decays faster than E_{j1}, and S_{jl} faster than $\partial_l E_{j1}$, in the sense of L^q-integrability. Thus the equations in (4) may in fact be considered as asymptotic expansions of u and ∇u, respectively. The theory in [10] is valid under the assumption that u verifies the boundary conditions

$$u(x) = e_1 + (\omega \times x) \quad \text{for } x \in \partial\mathfrak{D} \tag{6}$$

and f vanishes. Reference [10] does not deal with the pointwise decay of R and S.

In Theorem 3.1 below, we derive a pointwise decay of u and ∇u, respectively, which is independent of the boundary conditions, but compared to [10] and as indicated in (4) our leading term is less explicit than the term $\gamma\,E_{j1}(x)$ in (4), and, instead of the fundamental solution $E_{j1}(x)$ to the *stationary* Oseen system, we use the time integral of the fundamental solution to the *evolutionary* Oseen system.

In [5], it was shown that $\mathcal{Z}_{j1}(x,0) = E_{j1}(x)$ for $x \in \mathbb{R}^3\backslash\{0\}$, $1 \le j \le 3$, and $\lim_{|x|\to\infty} |\partial_x^\alpha \mathcal{Z}_{jk}(x,0)| = O\big((|x|\,s(x))^{-3/2-|\alpha|/2}\big)$ for $1 \le j \le 3$, $k \in \{2, 3\}$ [5, Corollary 4.5, Theorem 5.1]. Thus, setting

$$\mathfrak{G}_j(x) := \sum_{k=2}^{3} \beta_k\,\mathcal{Z}_{jk}(x,0) + \mathfrak{F}_j(x) \quad (x \in \overline{B_{S_1}}^c,\; 1 \le j \le 3), \tag{7}$$

we may deduce from (16) that