C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Calculus of variations

Minimizing movements along a sequence of functionals and curves of maximal slope

Mouvements minimisants le long d'une séquence de fonctionnelles et courbes de pente maximale

Andrea Braides [a], Maria Colombo [b], Massimo Gobbino [c], Margherita Solci [d]

[a] *Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy*
[b] *ETH Institute for Theoretical Studies, Clausiusstrasse 47, CH-8092 Zürich, Switzerland*
[c] *Dipartimento di Matematica, Via Filippo Buonarroti 1c, 56127 Pisa, Italy*
[d] *DADU, Università di Sassari, Piazza Duomo 6, 07041 Alghero (SS), Italy*

ARTICLE INFO

Article history:
Received 26 February 2016
Accepted after revision 25 April 2016
Available online xxxx

Presented by Haïm Brézis

ABSTRACT

We prove that a general condition introduced by Colombo and Gobbino to study limits of curves of maximal slope allows us to characterize also minimizing movements along a sequence of functionals as curves of maximal slope of a limit functional.

RÉSUMÉ

Nous montrons qu'une condition générale présentée par Colombo et Gobbino pour étudier les limites des courbes de pente maximale permet également de caractériser les mouvements minimisants le long d'une séquence de fonctionelles comme des courbes de pente maximale de la fonctionnelle limite.

1. Introduction

Following a vast earlier literature, the notion of minimizing movement has been introduced by De Giorgi to give a general framework for Euler schemes in order to define a gradient-flow-type motion also for a non-differentiable energy ϕ. It consists in introducing a time scale τ, defining a time-discrete motion by an iterative minimization procedure in which the distance from the previous step is penalized in a way depending on τ, and then obtaining a time-continuous limit as $\tau \to 0$. This notion is at the base of modern definitions of variational motion and has been successfully used to construct

E-mail addresses: braides@mat.uniroma2.it (A. Braides), maria.colombo@eth-its.ethz.ch (M. Colombo), massimo.gobbino@unipi.it (M. Gobbino), Margherita@uniss.it (M. Solci).

http://dx.doi.org/10.1016/j.crma.2016.04.011
1631-073X/

a theory of gradient flows in metric spaces by Ambrosio, Gigli, and Savaré [3]. In particular, under suitable assumptions, it can be shown that a minimizing movement is a curve of maximal slope for ϕ.

When a sequence of energies ϕ_ε parameterized by a (small positive) parameter ε has to be taken into account, in order to define an effective motion, one may examine the minimizing movements of ϕ_ε and take their limit as $\varepsilon \to 0$, or, depending on the problem at hand, instead compute the minimizing movement of the (Γ-)limit ϕ of ϕ_ε. In general, these two motions are different. This is due to the trivial fact that the energy landscape of ϕ may not carry enough information to describe the energy landscapes of ϕ_ε, since local minimizers may appear or disappear in the limit process; as easy examples, show [4].

A general approach is to proceed in the minimizing-movement scheme, letting the parameter ε and the time scale τ tend to 0 together. This gives a notion of *minimizing movement along ϕ_ε at a given scale τ*. In this way, we can detect fine phenomena due to the presence of local minima. The limit of the minimizing movements and the minimizing movement of the limit are recovered as extreme cases. A first example of this approach has been given in [6] for spin energies converging to a crystalline perimeter, in which case the extreme behaviours are complete pinning and flat flow [1]. For a general choice of the parameters, the limit motion is neither of the two, but depends on the ratio between ε and τ and is a degenerate motion by crystalline curvature with pinning only of large sets. More examples can be found in [4]. Note that in general the functions that we obtain as minimizing movements along a sequence cannot be easily rewritten as minimizing movements of a single functional.

In another direction, conditions have been exhibited that ensure that the limit of gradient flows for a family ϕ_ε, or of curves of maximal slope, be the gradient flow, or a curve of maximal slope, for their Γ-limit ϕ (see [8] and [7]). This suggests that under such conditions, all minimizing movements along ϕ_ε, at whatever scale, may converge to minimizing movements of ϕ. In this paper, we prove a result in that direction, showing that if a lower-semicontinuity inequality holds for the descending slope of ϕ_ε then any minimizing movement is a curve of maximal slope for ϕ (Theorem 2.3). This property holds in particular in the case of convex energies (see [7], and also [2] and [4]). Note that the limit curve of maximal slope may still depend on the way ε and τ tend to 0, and that in the extreme case of ε tending to 0 fast enough with respect to τ, it is also a minimizing movement for the limit (Theorem 2.5b). In the case where all curves of maximal slope are minimizing movements for ϕ, this is a kind of 'commutativity result' between minimizing movements and Γ-convergence. This again holds if ϕ is convex, which is automatic if also ϕ_ε are convex. Nevertheless, in some cases it has been possible to directly prove that minimizing movements along a sequence are minimizing movements for the limit, also for some non-convex energies as scaled Lennard–Jones ones [5].

The result presented in this note is suggested by the analog result for curves of maximal slope in [7]: it makes the analysis in [4] more precise, and we think it will be a useful reference for future applications. Its proof follows modifying the arguments of [3], which show that the minimizing movements for a single functional are curves of maximal slope, and is briefly presented at the end of Section 2.

2. The limit result

In what follows, (X, d) is a complete metric space.

Definition 2.1 *(Minimizing movements along ϕ_ε at scale τ_ε).* For all $\varepsilon > 0$ let $\phi_\varepsilon : X \to (-\infty, +\infty]$, and let $u^0_\varepsilon \in X$. Suppose that there exists some $\tau^* > 0$ such that for every $\varepsilon > 0$ and $\tau \in (0, \tau^*)$, there exists a sequence $\{u^i_{\varepsilon,\tau}\}$ that satisfies $u^0_{\varepsilon,\tau} = u^0_\varepsilon$ and $u^{i+1}_{\varepsilon,\tau}$ is a solution to the minimum problem

$$\min\Big\{\phi_\varepsilon(v) + \frac{1}{2\tau} d^2(v, u^i_{\varepsilon,\tau}) : v \in X\Big\}. \tag{1}$$

Let $\tau = \tau_\varepsilon$ be a family of positive numbers such that $\lim_{\varepsilon\to 0} \tau_\varepsilon = 0$ and define the piecewise-constant functions $\bar{u}_\varepsilon = \bar{u}_{\varepsilon,\tau_\varepsilon} : [0, +\infty) \to X$ as $\bar{u}_\varepsilon(t) = u^{i+1}_{\varepsilon,\tau}$ for $t \in (i\tau, (i+1)\tau]$. A *minimizing movement along ϕ_ε at scale τ_ε* with initial data u^0_ε is any pointwise limit of a subsequence of the family $\bar{u}_\varepsilon$.

From now on, we will make the following hypotheses, which ensure the existence of minimizing movements along the sequence ϕ_ε at any given scale τ_ε [4]: ($u^* \in X$ is an arbitrary given point):

(i) for all $\varepsilon > 0$, ϕ_ε is lower semicontinuous;

(ii) there exist $C^* > 0$ and $\tau^* > 0$ such that $\inf\big\{\phi_\varepsilon(v) + \frac{1}{2\tau^*} d(v, u^*) : v \in X\big\} \geq C^* > -\infty$ for all $\varepsilon > 0$;

(iii) for all $C > 0$, there exists a compact set K such that $\{u : d^2(u, u^*) \leq C, |\phi_\varepsilon(u)| \leq C\} \subset K$ for all $\varepsilon > 0$.

Remark 1. (a) Alternatively, in the definition above, we can suppose $\tau \to 0$ and choose $\varepsilon = \varepsilon_\tau \to 0$. In this way, we define *minimizing movements along ϕ_{ε_τ} at scale τ*;

(b) if $\phi_\varepsilon = \phi$ for all ε, then a minimizing movement along ϕ_ε at any scale is a (generalized) *minimizing movement for ϕ* as defined in [3].

Please cite this article in press as: A. Braides et al., Minimizing movements along a sequence of functionals and curves of maximal slope, C. R. Acad. Sci. Paris, Ser. I (2016), http://dx.doi.org/10.1016/j.crma.2016.04.011