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Algebraic geometry/Géométrie algébrique

Finiteness of Lagrangian fibrations with fixed invariants

Finitude des fibrations lagrangiennes à invariants fixes

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A R T I C L E I N F O

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A B S T R A C T

We prove finiteness of the deformation classes of hyperkähler Lagrangian fibrations in any fixed dimension with fixed Fujiki constant and discriminant of the Beauville–Bogomolov–Fujiki lattice. We also prove there are only finitely many deformation classes of hyperkähler Lagrangian fibrations with an ample line bundle of a given degree on the general fibre of the fibration.

R É S U M É

Nous démontrons la finitude des classes de déformation des fibrations lagrangiennes hyperkählériennes, de dimension quelconque, avec constante de Fujiki et discriminant du réseau de Beauville–Bogomolov–Fujiki fixes. Nous montrons également qu'il n'y a qu'un nombre fini de classes de déformation des fibrations lagrangiennes hyperkählériennes avec un fibré en droite ample de degré donné sur la fibre générale de la fibration.

1. Introduction

For a hyperkähler manifold M, the Fujiki constant and the discriminant of the Beauville–Bogomolov–Fujiki lattice are topological invariants. It is very natural to fix them and ask for finiteness of hyperkähler manifolds with these invariants. In this paper we establish finiteness of Lagrangian fibrations of hyperkähler manifolds with fixed topological invariants as above.

Theorem 1.1. *There are at most finitely many deformation classes of Lagrangian fibrations* $\pi : M \to \mathbb{C}P^n$ *with a fixed Fujiki constant* c *and a given discriminant of the Beauville–Bogomolov–Fujiki lattice* (Λ, q).

Francois Charles has the following boundedness result for families of hyperkähler varieties up to deformation. He drops the assumption that L is ample in Kollár–Matsusaka's theorem applied for hyperkähler manifolds and replaces it with the assumption that $q(L) > 0$.

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Theorem 1.2. *(See Charles [4].) Let n and r be two positive integers. Then there exists a scheme S of finite type over $\mathbb{C}$, and a projective morphism $\mathcal{M} \longrightarrow S$ such that if M is a complex hyperkähler variety of dimension 2n and L is a line bundle on M with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville–Bogomolov form, then there exists a complex point s of S such that $\mathcal{M}_s$ is birational to M.*

In our case, there is a natural line bundle L associated with the Lagrangian fibration. Using Fujiki's formula, it is a straightforward observation that $q(L) = 0$, while F. Charles deals with the case when $q(L) > 0$ (in which case M is projective by a result of D. Huybrechts: Theorem 3.11 in [8]).

In the proof of our main theorems we use F. Charles's finiteness result applied to an ample line bundle with minimal positive square of the Beauville–Bogomolov–Fujiki form. Since we are interested in a finiteness result up to deformation equivalence, one can obtain an ample line bundle after deforming a given Lagrangian fibration to a projective one. We also use lattice theory estimates applied to the Beauville–Bogomolov–Fujiki form.

In [13], Sawon proved a finiteness theorem for Lagrangian fibrations with a lot of natural assumptions on the fibration, such as existence of a section, fixed polarization type of a very ample line bundle, semi-simple degenerations as the general singular fibres, and a maximal variation of the fibres. We give the precise statement of Sawon's theorem in Section 2. Using the techniques in our proofs, one can also drop most of the other conditions in Sawon's theorem. We prove the following generalization.

Theorem 1.3. *Consider a Lagrangian fibration $\pi : M \longrightarrow \mathbb{C}P^n$ such that there is a line bundle P on M with $q(P) > 0$ and with a given P-degree d on the general fibre F of π, i.e., $P^n \cdot F = d$. Then there are at most finitely many deformation classes of hyperkähler manifolds M as above, i.e., they form a bounded family.*

For completeness of the exposition, we also mention Huybrechts' classical finiteness results.

Theorem 1.4. *(See Huybrechts [10].) If the second integral cohomology $H^2(\mathbb{Z})$ and the homogeneous polynomial of degree $2n - 2$ on $H^2(Z)$ defined by the first Pontrjagin class are given, then there exist at most finitely many diffeomorphism types of compact hyperkähler manifolds of real dimension 4n realizing this structure.*

Theorem 1.5. *(See Huybrechts [10].) Let M be a fixed compact manifold. Then there exist at most finitely many different deformation types of irreducible holomorphic symplectic complex structures on M.*

Using Theorem 1.5, the author and Misha Verbitsky established the following finiteness results in [11].

Theorem 1.6. *(See Kamenova–Verbitsky [11].) Let M be a fixed compact manifold. Then there are only finitely many deformation types of hyperkähler Lagrangian fibrations $(M, I) \longrightarrow \mathbb{C}P^n$, for all complex structures I on M.*

In the main theorem of this paper, we prove the finiteness of deformation classes of the total space M of the Lagrangian fibration $M \longrightarrow \mathbb{C}P^n$ with fixed dimension, Fujiki constant and discriminant of the Beauville–Bogomolov–Fujiki lattice. As a corollary of Theorem 1.6, one also obtains the finiteness of the deformation classes of the Lagrangian fibration $M \longrightarrow \mathbb{C}P^n$.

2. Hyperkähler geometry: preliminary results

Definition 2.1. A **hyperkähler manifold** is a compact Kähler holomorphic symplectic manifold.

Definition 2.2. A hyperkähler manifold M is called **simple** if $H^1(M, \mathbb{C}) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

Remark 2.3. From now on, we assume that all hyperkähler manifolds are simple.

Remark 2.4. The following two notions are equivalent: a holomorphic symplectic Kähler manifold and a manifold with a *hyperkähler structure*, that is, a triple of complex structures satisfying the quaternionic relations and parallel with respect to the Levi-Civita connection. In the compact case, the equivalence between these two notions is provided by Yau's solution to Calabi's conjecture [2]. In this paper, we assume compactness and we use the complex algebraic point of view.

Definition 2.5. Let M be a compact complex manifold and $\mathrm{Diff}^0(M)$ the connected component of the identity of its diffeomorphism group. Denote by Comp the space of complex structures on M, equipped with a structure of Fréchet manifold. The *Teichmüller space* of M is the quotient $\mathrm{Teich} := \mathrm{Comp} / \mathrm{Diff}^0(M)$. For a hyperkähler manifold M, the Teichmüller space is finite-dimensional [3]. Let $\mathrm{Diff}^+(M)$ be the group of orientable diffeomorphisms of a complex manifold M. The *mapping class group* $\Gamma := \mathrm{Diff}^+(M)/\mathrm{Diff}^0(M)$ acts naturally on Teich. For $I \in \mathrm{Teich}$, let Γ_I be the subgroup of Γ which fixes the connected component of complex structure I. The *monodromy group* is the image of Γ_I in $\mathrm{Aut}\, H^2(M, \mathbb{Z})$.

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