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Dynamical systems

Invariant measures for piecewise continuous maps

Mesures invariantes pour les applications continues par morceaux

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A R T I C L E I N F O

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A B S T R A C T

We say that $f : [0,1] \to [0,1]$ is a *piecewise continuous interval map* if there exists a partition $0 = x_0 < x_1 < \cdots < x_d < x_{d+1} = 1$ of $[0,1]$ such that $f|_{(x_{i-1},x_i)}$ is continuous and the lateral limits $w_0^+ = \lim_{x\to 0^+} f(x)$, $w_{d+1}^- = \lim_{x\to 1^-} f(x)$, $w_i^- = \lim_{x\to x_i^-} f(x)$ and $w_i^+ = \lim_{x\to x_i^+} f(x)$ exist for each i. We prove that every piecewise continuous interval map without connections admits an invariant Borel probability measure. We also prove that every injective piecewise continuous interval map with no connections and no periodic orbits is topologically semiconjugate to an interval exchange transformation.

R É S U M É

On dit que $f : [0,1] \to [0,1]$ est une *application d'intervalle continue par morceaux* s'il existe une partition $0 = x_0 < x_1 < \cdots < x_d < x_{d+1} = 1$ de $[0,1]$ telle que $f|_{(x_{i-1},x_i)}$ est continue et telle que les limites latérales $w_0^+ = \lim_{x\to 0^+} f(x)$, $w_{d+1}^- = \lim_{x\to 1^-} f(x)$, $w_i^- = \lim_{x\to x_i^-} f(x)$ et $w_i^+ = \lim_{x\to x_i^+} f(x)$ existent pour chaque i. On prouve que toute application d'intervalle continue par morceaux sans connexion admet une mesure de probabilité invariante. On prouve également que toute application injective d'intervalle continue par morceaux sans connexion et sans orbite périodique est topologiquement semiconjuguée à un échange d'intervalles.

1. Introduction

Much information about the long-term behaviour of the iterates of a map is revealed by its invariant measures. Regarding piecewise continuous interval maps, the presence of a non-atomic invariant Borel probability measure can be used to construct topological conjugacies or semiconjugacies with interval exchange transformations (IETs).

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Transfer operators have proved to be an important tool to obtain absolutely continuous invariant probability measures for piecewise smooth piecewise monotone interval maps (see [1,3–5,9]). In general, these types of results assume that each branch of the piecewise continuous map is C^r-smooth ($r \geq 1$), monotone and has derivative greater than 1.

The aim of this article is to prove the existence of invariant Borel probability measures for piecewise continuous interval maps not embraced by the transfer operator approach. In this way, our result includes gap maps, piecewise contractions and generalised interval exchange transformations (GIETs). No monotonicity and no smoothness assumptions, beyond the uniform continuity of each branch of the map, are assumed. Our result is the natural version of the Kryloff–Bogoliouboff Theorem (see [8]) for piecewise continuous interval maps.

We are also interested in constructing topological semiconjugacy between injective piecewise continuous interval maps and interval exchange transformations, possibly with flips. In this regard, it is worth mentioning the result by J. Milnor and W. Thurston (see [12]), which states that any continuous piecewise monotone interval map of positive entropy htop is topologically semiconjugate to a map with constant slope equal to $\pm e^{\mathrm{htop}}$. This result was generalised by L. Alsedà and M. Misiurewicz in [2] to piecewise continuous piecewise monotone interval maps of positive entropy. Concerning countably piecewise continuous piecewise monotone interval maps, a necessary and sufficient condition for the existence of a non-decreasing semiconjugacy to a map of constant slope was provided by M. Misiurewicz and S. Roth in [13]. The author and A. Nogueira proved in [14] that every injective piecewise contraction is topologically conjugate to a map with constant slope equal to $\pm\frac{1}{2}$.

The proof of the Kryloff–Bogoliouboff Theorem fails for discontinuous maps. In this article, we present a variation of this proof that overcomes such limitation. The hypothesis of no connections cannot be removed since there are examples of piecewise continuous maps that have connections and admit no Borel invariant measure. The proof presented here does not hold for countably piecewise continuous maps since for such maps the lateral limits might not exist at all points of $[0, 1]$.

2. Statement of the results

Throughout this article, assume that $f : [0, 1] \to [0, 1]$ is a piecewise continuous interval map. Hence, there exists a partition $0 = x_0 < x_1 < \cdots < x_d < x_{d+1} = 1$ of $[0, 1]$ such that $f|_{(x_{i-1},x_i)}$ is continuous and the lateral limits $w_0^+ = \lim_{x\to 0^+} f(x)$, $w_{d+1}^- = \lim_{x\to 1^-} f(x)$, $w_i^- = \lim_{x\to x_i^-} f(x)$ and $w_i^+ = \lim_{x\to x_i^+} f(x)$ exist for each i. Let

$$D = \{x_0, \ldots, x_{d+1}\}, \quad W = \{w_0^+, w_1^-, w_1^+, \ldots, w_d^-, w_d^+, w_{d+1}^-\}.$$

We say that f has *no connections* if

$$\bigcup_{w\in W} \bigcup_{k=0}^{\infty} \{f^k(w)\} \cap D = \emptyset. \tag{1}$$

We say that $x \in [0, 1]$ is a *periodic point* of f if there exists an integer $k \geq 1$ such that $f^k(x) = x$.

Our first result turns out to be a version of the Kryloff–Bogoliouboff Theorem [8] for piecewise continuous interval maps.

Theorem 2.1. *Let $f : [0, 1] \to [0, 1]$ be a piecewise continuous map with no connections, then f admits an invariant Borel probability measure μ. Moreover, if f has no periodic points, then the measure μ is non-atomic.*

The hypothesis of no connections in the statement of Theorem 2.1, although more readily checkable, may sound a bit restrictive because, for instance, it prohibits that a left-continuous map f takes one discontinuity into another. Indeed, what needs to be avoided for the existence of the invariant measure is the presence of *closed connections*, a more technical notion given in Section 3.

In the world of generalised interval exchange transformations, the hypothesis of no connections corresponds to the notion of having an ∞-complete path. As remarked in [11, p. 1586], every GIET with such property is topologically semiconjugate to an IET. The next result extends this claim to piecewise continuous maps. It can also be considered a generalisation of the item (a) of the Structure Theorem by Gutierrez [6, p. 18].

Corollary 2.2. *Let $f : [0, 1] \to [0, 1]$ be an injective piecewise continuous map with no connections and no periodic points, then f is topologically semiconjugate to an interval exchange transformation, possibly with flips.*

Now we present a class of piecewise continuous interval maps for which having no connections is a generic (in the measure-theoretical sense) property. We recall that an irrationality criterion for the absence of connections in IETs without flips was provided by M. Keane in [7].

Theorem 2.3. *Let $\phi_1, \ldots, \phi_{d+1} : [0, 1] \to (0, 1)$ be continuous maps and let $\Omega \subset \mathbb{R}^d$ be the open set $\Omega = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < \cdots < x_d < 1\}$, then for Lebesgue almost every $(x_1, \ldots, x_d) \in \Omega$, the piecewise continuous map $f : [0, 1] \to (0, 1)$ defined by $f(x) = \phi_i(x)$ if $x \in I_i$, where $I_1 = [0, x_1)$, $I_2 = [x_1, x_2), \ldots, I_d = [x_{d-1}, x_d)$, $I_{d+1} = [x_d, 1]$, has no connections and hence admits an invariant Borel probability measure.*

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