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Numerical analysis

A new fictitious domain method: Optimal convergence without cut elements

Une nouvelle méthode de type domaine fictif : convergence optimale sans éléments coupés

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A R T I C L E   I N F O

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A B S T R A C T

We present a method of the fictitious domain type for the Poisson–Dirichlet problem. The computational mesh is obtained from a background (typically uniform Cartesian) mesh by retaining only the elements intersecting the domain where the problem is posed. The resulting mesh does not thus fit the boundary of the problem domain. Several finite element methods (XFEM, CutFEM) adapted to such meshes have been recently proposed. The originality of the present article consists in avoiding integration over the elements cut by the boundary of the problem domain, while preserving the optimal convergence rates, as confirmed by both the theoretical estimates and the numerical results.

R É S U M É

Nous présentons une méthode de type domaine fictif pour le problème de Poisson–Dirichlet. Le maillage de calcul est construit à partir d'un maillage ambiant (typiquement uniforme cartésien) en rejetant les éléments en dehors du domaine dans lequel le problème est posé. Le maillage ainsi obtenu n'est pas ajusté à la frontière du domaine du problème. Plusieurs méthodes d'éléments finis (XFEM, CutFEM) adaptées à ce type de maillages ont été proposées récemment. L'originalité de la méthode que l'on propose ici réside dans le fait que l'on évite l'intégration sur les éléments coupés par la frontière du domaine du problème, tout en préservant le taux de convergence optimal. Cette observation est confirmée par une étude théorique et par des essais numériques.

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1. Introduction and presentation of the method

Consider the Poisson problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = g \text{ on } \Gamma \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain with smooth boundary Γ, f and g are given functions on Ω and Γ respectively. The goal of the article is to construct a fictitious domain finite element (FE) discretization of problem (1) whose convergence rate is the same as that of a standard FE discretization on a mesh fitting the geometry of Ω. We start by embedding Ω into a simply shaped domain $\mathcal{O}$ and introduce a quasi-uniform mesh $\mathcal{T}_h^{\mathcal{O}}$ on $\mathcal{O}$ that can cut the boundary Γ in an arbitrary manner. Let

$$\mathcal{T}_h = \{T \in \mathcal{T}_h^{\mathcal{O}} : T \cap \Omega \neq \varnothing\}, \qquad \Omega_h = \left(\cup_{T\in\mathcal{T}_h} T\right)^\circ$$

$\Gamma_h = \partial\Omega_h$, as illustrated in Fig. 1. Several optimally convergent fictitious domain methods have been recently proposed following the XFEM or CutFEM paradigm. The FE approximation to u is sought there in a FE space defined over the mesh $\mathcal{T}_h$ and boundary conditions on Γ are imposed either through Lagrange multipliers [2,5] or by the Nitsche method [1,3]. The common feature of all these methods is that the integrals over Ω are preserved in the FE formulation so that a non-trivial numerical quadrature should be performed to compute the contributions to the stiffness matrix and to the right-hand side on the parts of mesh elements obtained by cutting $\mathcal{T}_h$ with Γ. We attempt, in the present paper, to circumvent this technical complication by introducing a reformulation of the problem that involves the integrals only over Ω_h, Γ_h, and Γ.

Let us extend f from Ω to Ω_h and imagine (for the moment) that (1) can be solved on the extended domain Ω_h while still imposing the boundary conditions on Γ:

$$-\Delta u = f \text{ in } \Omega_h, \qquad u = g \text{ on } \Gamma. \tag{2}$$

We keep here the same notations u and f for the functions on Ω_h as for the originals on Ω. Integration by parts over Ω_h yields

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} u \frac{\partial v}{\partial n} + \frac{\gamma}{h}\int_{\Gamma} uv = \int_{\Omega_h} fv + \int_{\Gamma} g\frac{\partial v}{\partial n} + \frac{\gamma}{h}\int_{\Gamma} gv \tag{3}$$

for any $v \in H^1(\Omega_h)$ and $\gamma > 0$. Here, n on Γ or Γ_h denotes the unit normal looking outwards from Ω or Ω_h.

We inspire ourselves with the variational formulation (3) in writing the following FE discretization: introduce

$$V_h = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_1(T)\, \forall T \in \mathcal{T}_h\}$$

with $\mathbb{P}_1$ denoting the set of polynomials of degree ≤ 1 and search for $u_h \in V_h$ such that

$$a_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h \tag{4}$$

where

$$a_h(u,v) = \int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} u\frac{\partial v}{\partial n} + \frac{\gamma}{h}\int_{\Gamma} uv + \sigma h \sum_{E\in\mathcal{F}_\Gamma} \int_E \left[\frac{\partial u}{\partial n}\right]\left[\frac{\partial v}{\partial n}\right]$$

$$L_h(v) = \int_{\Omega_h} fv + \int_{\Gamma} g\frac{\partial v}{\partial n} + \frac{\gamma}{h}\int_{\Gamma} gv,$$

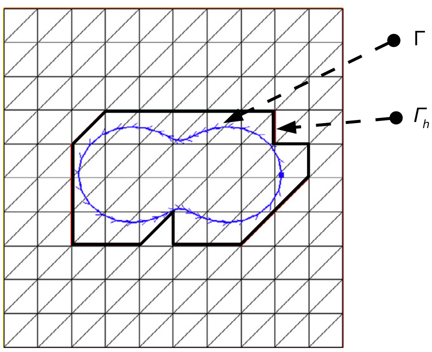

Fig. 1. The "background" mesh $\mathcal{T}_h^{\mathcal{O}}$, the "physical" domain Ω (inside Γ) and the computational domain Ω_h (inside Γ_h).