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Combinatorics

Trimness of closed intervals in Cambrian semilattices ☆

Sveltesse des intervalles bornés d'un demi-treillis cambrien

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A R T I C L E   I N F O

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A B S T R A C T

In this article, we give a short algebraic proof that all closed intervals in a γ-Cambrian semilattice $\mathcal{C}_\gamma$ are trim for any Coxeter group W and any Coxeter element $\gamma \in W$. This means that if such an interval has length k, then there exists a maximal chain of length k consisting of left-modular elements, and there are precisely k join- and k meet-irreducible elements in this interval. Consequently, every graded interval in $\mathcal{C}_\gamma$ is distributive. This problem was open for any Coxeter group that is not a Weyl group.

R É S U M É

Dans cet article, nous donnons une démonstration courte et algébrique du fait que tous les intervalles bornés d'un demi-treillis γ-cambrien $\mathcal{C}_\gamma$ sont sveltes pour tout groupe de Coxeter W et tout élément de Coxeter $\gamma \in W$. Cela signifie que, si un tel intervalle a pour longueur k, il existe une chaîne de longueur k consistant en éléments modulaires à gauche, et il y a exactement k éléments sup-irréductibles et k éléments inf-irréductibles. En conséquence, il s'ensuit que chaque intervalle gradué est distributif. Ce problème était ouvert pour tout groupe de Coxeter qui n'est pas un groupe de Weyl.

1. Introduction

The γ-Cambrian semilattice, denoted by $\mathcal{C}_\gamma$ and parameterized by a Coxeter group W and a Coxeter element $\gamma \in W$, was introduced by N. Reading and D. Speyer in [16], generalizing N. Reading's construction for finite W [14,15]. This family of semilattices can be seen as a generalization of the famous Tamari lattices to all Coxeter groups, in the sense that the γ-Cambrian semilattice associated with the symmetric group $\mathfrak{S}_n$ and the Coxeter element $\gamma = (1\,2\,\ldots\,n)$ is isomorphic to the Tamari lattice of parameter n. The Tamari lattices play an important role in algebraic and geometric combinatorics, and frequently occur in many seemingly unrelated branches of mathematics [11].

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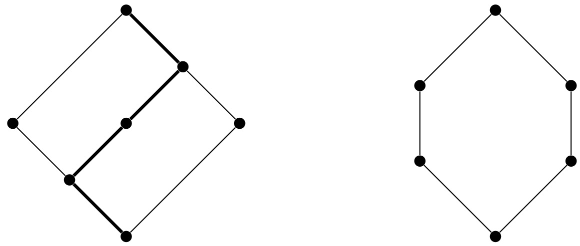

(a) A trim lattice that is not semidistributive.

(b) A semidistributive lattice that is not trim.

Fig. 1. Different generalizations of distributivity.

Previously, many of the properties of the Tamari lattices have been generalized to the γ-Cambrian semilattices, such as EL-shellability and the topology of their order complexes [8,12], congruence-uniformity [13,16,17], or semidistributivity [16]. Another property that the Tamari lattices enjoy is trimness, introduced by H. Thomas [18]. A finite lattice of length k is *trim* if it possesses a left-modular chain of length k, and if it has precisely k join- and k meet-irreducible elements. An interesting observation that goes back to G. Markowsky [10] is the fact that graded trim lattices are distributive. Hence trimness can be seen as a generalization of distributivity to ungraded lattices. It was conjectured in [18] that $\mathcal{C}_\gamma$ is trim for any finite Coxeter group W, and any Coxeter element $\gamma \in W$. In the same paper, it was shown that this conjecture is true when W is of type A or B, using the permutation representation of these groups and the definition of $\mathcal{C}_\gamma$ as posets on certain pattern-avoiding permutations [18, Theorems 8 and 9]. Subsequently, it was shown in [7, Theorem 4.17] that the said conjecture holds when W is a Weyl group, using the definition of $\mathcal{C}_\gamma$ as posets on torsion classes of certain quiver representations. It is the purpose of this paper to prove the trimness of $\mathcal{C}_\gamma$ in full generality, namely when W is an arbitrary (perhaps infinite) Coxeter group, and $\gamma \in W$ is any Coxeter element. More precisely, we prove the following theorem.

Theorem 1.1. *Every closed interval in a γ-Cambrian semilattice $\mathcal{C}_\gamma$ is trim for any Coxeter group W and any Coxeter element $\gamma \in W$. In particular, every graded closed interval in $\mathcal{C}_\gamma$ is distributive.*

Recall that when W is infinite, $\mathcal{C}_\gamma$ is only a semilattice since it does not possess a maximal element. However, any closed interval in $\mathcal{C}_\gamma$ is a finite lattice in its own right, and Theorem 1.1 thus implies the local trimness of $\mathcal{C}_\gamma$.

The proof of Theorem 1.1 uses the definition of $\mathcal{C}_\gamma$ in terms of sortable elements, and is thus uniform. One ingredient for proving Theorem 1.1 is the semidistributivity of $\mathcal{C}_\gamma$ that was established in [16, Section 8]. Semidistributivity can be seen as another generalization of distributivity to ungraded lattices, but it is different from trimness. Consider for instance the lattice in Fig. 1(a). This is a trim lattice, since it has length 4, it has exactly four join- and meet-irreducible elements, and the highlighted chain consists of left-modular elements. However, it is not semidistributive, since it is precisely one of the minimal lattices that were used in [2] to characterize the obstructions to semidistributivity. Conversely, Fig. 1(b) shows a semidistributive lattice that is not trim, since it has four join- and four meet-irreducible elements, but length only three.

We recall the necessary background on Coxeter groups in Section 2.1, and recall the definition of the γ-Cambrian semilattices in Section 2.2. We define the non-standard poset-theoretical concepts whenever needed, and refer to [3] for any undefined terminology. In Section 3, we prove Theorem 1.1.

2. Preliminaries

In this section, we give the definitions needed in the remainder of this article. For more information on Coxeter groups, we refer to [1] and [6]. An excellent exposition on γ-Cambrian semilattices is [16]. Throughout the article we use the abbreviation $[n] = \{1, 2, \ldots, n\}$.

2.1. Coxeter groups

A *Coxeter system* (W, S) is a pair given by a group W (whose identity is denoted by ε) and a subset $S \subseteq W$ with $S = \{s_1, s_2, \ldots, s_n\}$ such that W admits the presentation

$$W = \big\langle S \mid (s_i s_j)^{m_{i,j}} = \varepsilon \text{ for } i, j \in [n] \big\rangle.$$

In this presentation, each $m_{i,j}$ is either a positive integer or the formal symbol ∞, and these numbers satisfy $m_{i,j} = m_{j,i} \geq 1$ for $i, j \in [n]$, and $m_{i,j} = 1$ if and only if $i = j$. In particular, $m_{i,j} = \infty$ means that there is no relation between the generators s_i and s_j. The elements $s_1, s_2, \ldots, s_n$ are called the *simple generators* of W and n is called the *rank* of W. Since S generates the group W, every $w \in W$ can be written as a product of those simple generators. This gives rise to the length function

$$\ell_S(w) = \min\big\{k \mid w = s_{i_1} s_{i_2} \cdots s_{i_k}, \text{ where } s_{i_j} \in S \text{ for } j \in [k]\big\}.$$

Now we can define the *(right) weak order* on W by