C. R. Acad. Sci. Paris, Ser. I 354 (2016) 125–130

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Combinatorics/Mathematical physics

Tropical curves in sandpiles ☆

Nikita Kalinin, Mikhail Shkolnikov

Université de Genève, Section de mathématiques, route de Drize 7, villa Battelle, 1227 Carouge, Switzerland

A R T I C L E I N F O

Article history:
Received 18 August 2015
Accepted after revision 6 November 2015
Available online 26 January 2016

Presented by the Editorial Board

Keywords:
Combinatorics
Mathematical physics

A B S T R A C T

We study a sandpile model on the set of the lattice points in a large lattice polygon. A small perturbation ψ of the maximal stable state $\mu \equiv 3$ is obtained by adding extra grains at several points. It appears that the result ψ° of the relaxation of ψ coincides with μ almost everywhere; the set where $\psi^\circ \neq \mu$ is called the deviation locus. The scaling limit of the deviation locus turns out to be a distinguished tropical curve passing through the perturbation points.

R É S U M É

Nous considérons le modèle du tas de sable sur l'ensemble des points entiers d'un polygone entier. En ajoutant des grains de sable en certains points, on obtient une perturbation mineure de la configuration stable maximale $\mu \equiv 3$. Le résultat ψ° de la relaxation est presque partout égal à μ. On appelle lieu de déviation l'ensemble des points où $\psi^\circ \neq \mu$. La limite au sens de la distance de Hausdorff du lieu de déviation est une courbe tropicale spéciale, qui passe par les points de perturbation.

1. Introduction

1.1. Sandpile model on a finite set

Consider the standard lattice $\mathbb{Z}^2$ on the plane. For $v \in \mathbb{Z}^2$ we denote by $n(v)$ the set of all four closest points to v in $\mathbb{Z}^2$. Let Γ be a finite subset of $\mathbb{Z}^2$. A *state* (or a *configuration*) $\phi \in \mathbb{Z}^\Gamma$ of a sandpile on Γ is a non-negative integer-valued function on Γ. For a state ϕ and $v \in \Gamma$, we interpret $\phi(v)$ as a number of sand grains at the site v. For each $v \in \Gamma$, we define the *toppling* operator $T_v : \mathbb{Z}^\Gamma \to \mathbb{Z}^\Gamma$ at v given by

$$T_v \phi = \phi - 4\delta_v + \sum_{w \in \Gamma \cap n(v)} \delta_w,$$

where δ_v is a function on the lattice defined to be 1 at v and 0 otherwise. A toppling T_v is called *legal* for a state ϕ if $\phi(v) \geq 4$, i.e. if $T_v \phi$ is again a state. In this case, we think of T_v as a redistribution of sand from the overfilled site v to its

☆ Research is supported in part the grant 159240 of the Swiss National Science Foundation as well as by the National Center of Competence in Research SwissMAP of the Swiss National Science Foundation.

E-mail addresses: Nikita.Kalinin@unige.ch (N. Kalinin), mikhail.shkolnikov@gmail.com (M. Shkolnikov).

http://dx.doi.org/10.1016/j.crma.2015.11.003
1631-073X/

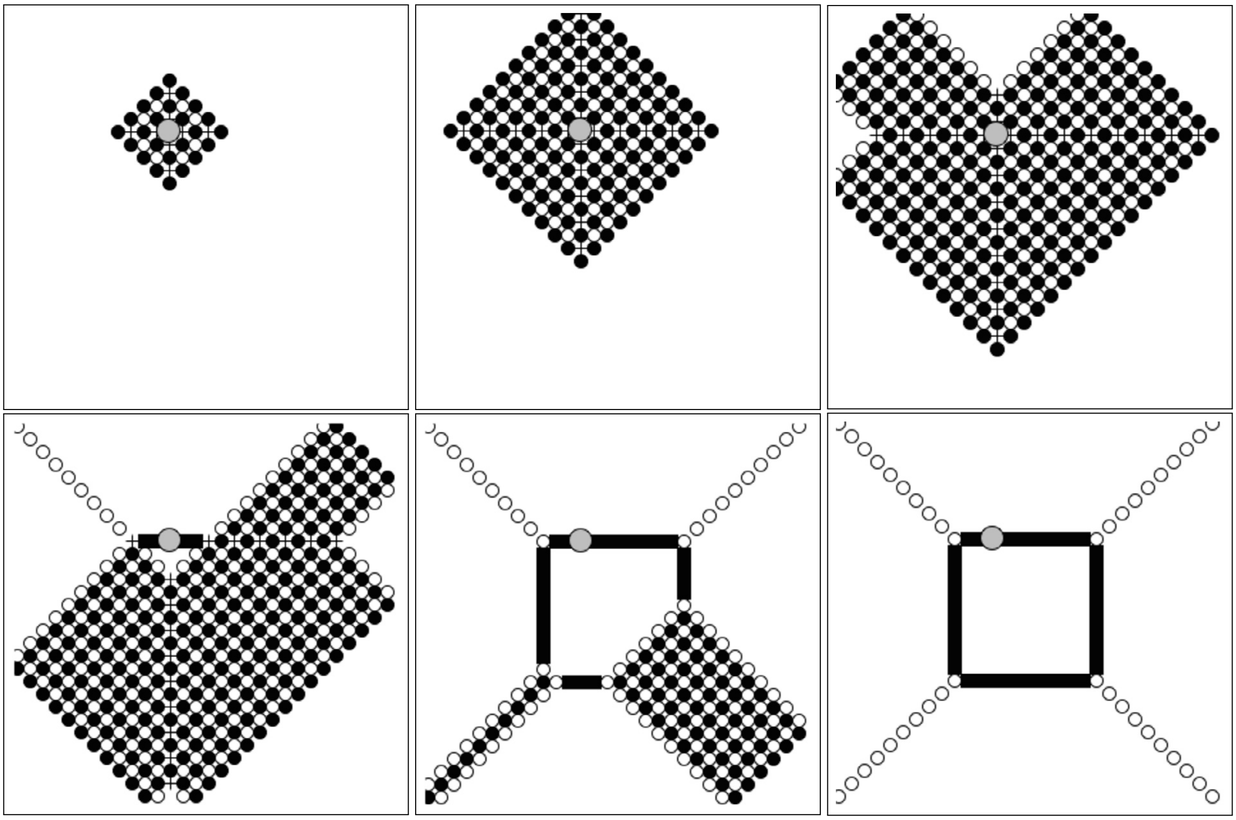

Fig. 1. Snapshots during the relaxation for the state $\phi \equiv 3$ on a square after adding an extra grain at one point p (the big grey point). Black rounds represent x with $\phi(x) \geq 4$, black squares (which are arranged along the vertical and horizontal edges on the final picture) represent the value of sand equal to 2, white rounds (arranged along diagonals on the final picture) are 1, and whites cells are 3. Rare cells with zero grains are marked as crosses, one can see them during the relaxation on the vertical and horizontal lines through p. The value of the final state at p is 3.

Fig. 1. Instantanés pendant la relaxation de la configuration $\phi \equiv 3$ sur un carré, après ajout d'un grain additionnel au point p (le gros point gris). Les ronds noirs représentent x avec $\phi(x) \geq 4$, les carrés noirs (qui se trouvent sur les arêtes verticales et horizontales dans la configuration finale) représentent les points où le nombre de grains de sable ést egal à 2. Les cercles (sur les diagonales dans la configuration finale) possèdent un grain de sable et les cellules blanches en ont 3. Les cellules rares avec zéro grain sont marquées par des croix ; on peut les voir pendant la relaxation sur les lignes verticales et horizontales passant par p. La valeur au point p dans la configuration finale est 3.

neighbors. If some neighbors are missing in Γ, i.e. $n(v) \not\subset \Gamma$, then at least one grain leaves the system after the toppling. If $\phi(v) < 4$ for all $v \in \Gamma$, then ϕ is called a *stable* state. The state μ, which is defined to be equal to 3 at every point of Γ, is called the *maximal stable state*.

A *relaxation* for a state ϕ is a sequence of states $\phi = \phi_0, \phi_1, \dots \phi_m$ such that ϕ_{i+1} is the result of applying a legal toppling to ϕ_i and ϕ_m is a stable state. It is well known that, for any state ϕ, there exists a relaxation and the last state ϕ_m depends only on ϕ (see [1,5]). We denote ϕ_m by ϕ° and call it *the result of the relaxation* of ϕ. Informally, in order to find the result of the relaxation, it does not matter which legal topplings are applied at each step of a relaxation sequence.

1.2. Motivation

Let $\Omega \subset \mathbb{R}^2$ be a non-degenerate (i.e. of non-zero area) lattice polygon. Let Γ be the intersection of $\mathbb{Z}^2$ with Ω. Consider a set $P \subset \Gamma$. We add extra grains to the state μ at the points $P = \{p_1, p_2, \dots, p_n\}$. After the relaxation, this gives the state $\phi = (3 + \sum_{p \in P} \delta_p)^\circ$.

In the examples shown in Figs. 1, 2 and 3, we see that the set of points where ϕ is not maximal constitutes some sort of a graph passing through P. As it can be seen from Figs. 2 and 3, the picture becomes more regular when the cardinality of P is small with respect to the size of Ω. In the next section, we state certain theorems formalizing this concept. Some of this results and their far-going generalizations will appear in [7]. This short note can be seen as an introduction to the subject and an announcement of these results.