C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

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C. R. Acad. Sci. Paris, Ser. I

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Strong convergence in the weighted setting of operator-valued Fourier series defined by the Marcinkiewicz multipliers

Fonctions de la classe de Marcinkiewicz et la convergence forte des séries d'opérateurs de Fourier associées

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A R T I C L E I N F O

Article history:
Received 4 July 2015
Accepted 4 November 2015
Available online xxxx

Presented by Gilles Pisier

Keywords:
Marcinkiewicz class
Fourier multiplier
Fourier series
Modulus mean-bounded operator
Shift operator
A_p weight sequence

A B S T R A C T

Suppose that $1 < p < \infty$ and let w be a bilateral weight sequence satisfying the discrete Muckenhoupt A_p weight condition. We show that every Marcinkiewicz multiplier $\psi : \mathbb{T} \to \mathbb{C}$ has an associated operator-valued Fourier series which serves as an analogue in $\mathfrak{B}\left(\ell^p(w)\right)$ of the usual Fourier series of ψ, and this operator-valued Fourier series is everywhere convergent in the strong operator topology. In particular, we deduce that the partial sums of the usual Fourier series of ψ are uniformly bounded in the Banach algebra of Fourier multipliers for $\ell^p(w)$. These results transfer to the framework of invertible, modulus mean-bounded operators acting on L^p spaces of sigma-finite measures.

R É S U M É

Soient $1 < p < \infty$ et w un poids dans la classe $A_p(\mathbb{Z})$. Cette note établit (dans la topologie forte des opérateurs) la convergence des séries de Fourier (à valeurs dans $\mathfrak{B}\left(\ell^p(w)\right)$) pour les « convolutions de Stieltjes », où ces convolutions sont déterminées par les fonctions ψ appartenant à la classe de Marcinkiewicz $\mathfrak{M}_1(\mathbb{T})$. Les propriétés de convergence pour ces séries de Fourier ayant valeurs dans $\mathfrak{B}\left(\ell^p(w)\right)$ révèlent des propriétés de convergence des séries de Fourier traditionnelles pour les fonctions $\psi \in \mathfrak{M}_1(\mathbb{T})$. En particulier, les sommes partielles de la série de Fourier traditionnelle pour un $\psi \in \mathfrak{M}_1(\mathbb{T})$ quelconque sont uniformément bornées dans la norme des p-multiplicateurs pour $\ell^p(w)$. Ces résultats se transfèrent immédiatement au cadre d'une bijection linéaire arbitraire T telle que T soit un opérateur préservant la disjonction dont le module linéaire est à moyennes bornées.

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http://dx.doi.org/10.1016/j.crma.2015.11.001

1. Introduction

The symbol K with (a possibly empty) set of subscripts denotes a constant which depends only on those subscripts, and which may change in value from one occurrence to another. The characteristic function of an arc $\mathfrak{A} \subseteq \mathbb{T}$ will be symbolized by $\chi_{\mathfrak{A}}$. For our treatment of Marcinkiewicz multipliers we shall make free use of the standard notation for the sequence $\{t_k\}_{k=-\infty}^{\infty}$ of dyadic points of the interval (0.2π), which are defined as $2^{k-1}\pi$ if $k \le 0$, and $2\pi - 2^{-k}\pi$ if $k > 0$. For $1 < p < \infty$, a weight sequence $w \equiv \{w_k\}_{k=-\infty}^{\infty}$ belongs to the class $A_p(\mathbb{Z})$ provided that there is a real constant C (called an $A_p(\mathbb{Z})$ *weight constant for* w) such that

$$\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k\right)\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k^{-1/(p-1)}\right)^{p-1} \le C,$$

whenever $L \in \mathbb{Z}$, $M \in \mathbb{Z}$, and $L \le M$. We denote the corresponding sequence space by $\ell^p(w)$. We say that $\psi \in L^\infty(\mathbb{T})$ is a *multiplier for* $\ell^p(w)$ (in symbols, $\psi \in M_{p,w}(\mathbb{T})$) provided that convolution by the inverse Fourier transform of ψ defines a bounded operator on $\ell^p(w)$. Specifically, we require:

Definition 1.1.

(i) For each $x \equiv \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$ and each $j \in \mathbb{Z}$, the series

$$\left(\psi^\vee * x\right)(j) \equiv \sum_{k=-\infty}^{\infty} \psi^\vee(j-k)\, x_k \text{ converges absolutely, and}$$

(ii) the mapping $\mathcal{T}_\psi^{(p,w)} : x \in \ell^p(w) \to \psi^\vee * x$ is a bounded linear mapping of $\ell^p(w)$ into $\ell^p(w)$.

We then call $\mathcal{T}_\psi^{(p,w)}$ the multiplier transform corresponding to ψ, and define the multiplier norm by setting $\|\psi\|_{M_{p,w}(\mathbb{T})} \equiv \left\|\mathcal{T}_\psi^{(p,w)}\right\|_{\mathfrak{B}(\ell^p(w))}$. In particular, it is well-known that $\mathfrak{M}_1(\mathbb{T}) \subseteq M_{p,w}(\mathbb{T})$, where $\mathfrak{M}_1(\mathbb{T})$ is the Banach algebra of periodic Marcinkiewicz multipliers, consisting of all functions $\psi \colon \mathbb{T} \to \mathbb{C}$ such that

$$\|\psi\|_{\mathfrak{M}_1(\mathbb{T})} \equiv \sup_{z\in\mathbb{T}} |\psi(z)| + \sup_{k\in\mathbb{Z}} \operatorname{var}(\psi, \Delta_k) < \infty$$

(here Δ_k is the dyadic arc of $\mathbb{T}$ specified by $\Delta_k = \left\{e^{i\theta} : \theta \in [t_k, t_{k+1}]\right\}$). Moreover, $\|\psi\|_{M_{p,w}(\mathbb{T})} \le K_{p,C}\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}$. A key structural example of an element of $M_{p,w}(\mathbb{T})$ is furnished, for each $k \in \mathbb{Z}$, by the function $\mathfrak{e}_k(z) \equiv z^k$, whose multiplier transform is $\mathcal{L}^k$, where $\mathcal{L}$ designates the left bilateral shift on $\ell^p(w)$. In particular, for each $\phi \in L^1(\mathbb{T})$, the $n^{\underline{th}}$ partial sum of its Fourier series $s_n\left(\phi, e^{i\theta}\right) \equiv \sum_{k=-n}^{n} \widehat{\phi}(k)\, e^{ik\theta}$ belongs to $M_{p,w}(\mathbb{T})$, with multiplier transform expressed by

$$\mathcal{T}_{s_n(\phi,\cdot)}^{(p,w)} = \sum_{k=-n}^{n} \widehat{\phi}(k)\, \mathcal{L}^k.$$

For further background items concerning our framework, the reader is referred to [1–3]. Our main result can now be stated as follows.

Theorem 1.2. *Suppose that* $\psi \in \mathfrak{M}_1(\mathbb{T})$. *Then whenever* $1 < p < \infty$, *and* $w \in A_p(\mathbb{Z})$ *with an* $A_p(\mathbb{Z})$ *weight constant* C, *we have:*

$$\sup\left\{\|s_n(\psi_z, (\cdot))\|_{M_{p,w}(\mathbb{T})} : n \ge 0, z \in \mathbb{T}\right\} \le K_{p,C}\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}, \tag{1.1}$$

where $\psi_z(\cdot) \equiv \psi((\cdot)z)$. *Consequently,* $\sum_{k=-\infty}^{\infty} z^k\, \widehat{\psi}(k)\, \mathcal{L}^k$, *the Fourier series of* $\mathcal{T}_{\psi_z}^{(p,w)}$ *relative to the strong operator topology of* $\mathfrak{B}\left(\ell^p(w)\right)$, *converges in the strong operator topology to* $\mathcal{T}_{\psi_z}^{(p,w)}$ *at each* $z \in \mathbb{T}$.

Thanks to the Dominated Ergodic Estimate Theorem of F.J. Martín-Reyes and A. de la Torre (in the form and notation of Theorem 2.5 in [3]), one can transfer Theorem 1.2 to a broader framework, where the following outcome ensues.

Theorem 1.3. *Suppose that* $1 < p < \infty$, (Ω, μ) *is a sigma-finite measure space, and* $\mathfrak{U} \in \mathfrak{B}\left(L^p(\mu)\right)$ *is an invertible, disjoint, modulus mean-bounded operator. Let* $\mathcal{E}(\cdot) : \mathbb{R} \to \mathfrak{B}\left(L^p(\mu)\right)$ *be the (idempotent-valued) spectral decomposition of* $\mathfrak{U}$, *and let* $\psi \in \mathfrak{M}_1(\mathbb{T})$ *be a continuous function. Then*

Please cite this article in press as: E. Berkson, Strong convergence in the weighted setting of operator-valued Fourier series defined by the Marcinkiewicz multipliers, C. R. Acad. Sci. Paris, Ser. I (2015), http://dx.doi.org/10.1016/j.crma.2015.11.001