C. R. Acad. Sci. Paris, Ser. I 354 (2016) 7–11

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical analysis

Some simple conditions for univalence

Quelques conditions simples pour l'univalence

Mamoru Nunokawa [a], Janusz Sokół [b]

[a] *University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan*
[b] *Department of Mathematics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland*

ARTICLE INFO

Article history:
Received 8 July 2015
Accepted after revision 5 October 2015
Available online 27 October 2015

Presented by the Editorial Board

Keywords:
Analytic
Univalent
Convex
Starlike
Close-to-convex
Differential subordination

ABSTRACT

We apply Ozaki–Umezawa's lemma on functions that are convex in one direction to find some sufficient conditions for univalence and closeness to convexity.

RÉSUMÉ

Nous appliquons le lemme de Ozaki et Umezawa sur les fonctions convexes dans une direction, afin de trouver des conditions suffisantes pour l'univalence et la presque convexité.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{D}$ and normalized, i.e. $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$. We say that $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disk $\mathbb{D}$, written $f \prec g$ if and only if there exists an analytic function $w \in \mathcal{H}$ such that $|w(z)| \le |z|$ and $f(z) = g[w(z)]$ for $z \in \mathbb{D}$. Therefore $f \prec g$ in $\mathbb{D}$ implies $f(\mathbb{D}) \subset g(\mathbb{D})$. In particular if g is univalent in $\mathbb{D}$, then the Subordination Principle says that $f \prec g$ if and only if $f(0) = g(0)$ and $f(|z| < r) \subset g(|z| < r)$, for all $r \in (0, 1]$.

Let us recall the Ozaki–Umezawa's lemma [6,8].

Lemma 1.1. *Let $f(z) = z + a_2 z^2 + \cdots$ be analytic for $|z| \le 1$ and $f'(z) \ne 0$ on $|z| = 1$. If there holds the relation*

$$\int_0^{2\pi} \left| 1 + \Re\mathfrak{e} \left\{ \frac{zf''(z)}{f'(z)} \right\} \right| d\theta < 4\pi, \quad |z| = 1, \tag{1.1}$$

then $f(z)$ is convex in one direction and hence $f(z)$ is univalent in $|z| \le 1$.

E-mail addresses: mamoru_nuno@doctor.nifty.jp (M. Nunokawa), jsokol@prz.edu.pl (J. Sokół).

http://dx.doi.org/10.1016/j.crma.2015.10.001
1631-073X/

2. Main result

Theorem 2.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. Assume that*

$$\left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \le |1 + \mathfrak{Re}\{\alpha_0 z\}| \quad (z \in \mathbb{D}), \tag{2.1}$$

where $\alpha_0 = 1/\cos t_0$ and t_0 is the positive root of the equation

$$\tan t = t + \pi/2, \quad 0 < t < \pi/2. \tag{2.2}$$

Then $f(z)$ is univalent in $\mathbb{D}$. Note that and $2.909 < \alpha_0 < 2.992$.

Proof. Applying [1] and [4], we have from (2.1)

$$\begin{aligned}\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta &= \int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right| \mathrm{d}\theta \\ &\le \int_0^{2\pi} \left|1 + \mathfrak{Re}\left\{\alpha_0 re^{i\theta}\right\}\right| \mathrm{d}\theta,\end{aligned}$$

where $0 < r < 1$. Letting $r \to 1$, we have

$$\begin{aligned}&\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta \\ &\le \int_0^{2\pi} |1 + \alpha_0 \cos\theta| \, \mathrm{d}\theta.\end{aligned}$$

We have

$$\cos^{-1}(-1/\alpha_0) = \pi - \cos^{-1}(1/\alpha_0). \tag{2.3}$$

Thus we obtain

$$\begin{aligned}&\int_0^{2\pi} |1 + \alpha_0 \cos\theta| \, \mathrm{d}\theta \\ &= 2 \int_0^{\cos^{-1}(-1/\alpha_0)} (1 + \alpha_0 \cos\theta) \, \mathrm{d}\theta - \int_{\cos^{-1}(-1/\alpha_0)}^{2\pi - \cos^{-1}(-1/\alpha_0)} (1 + \alpha_0 \cos\theta) \, \mathrm{d}\theta \\ &= 2\left[\theta + \alpha_0 \sin\theta\right]_0^{\cos^{-1}(-1/\alpha_0)} - \left[\theta + \alpha_0 \sin\theta\right]_{\cos^{-1}(-1/\alpha_0)}^{2\pi - \cos^{-1}(-1/\alpha_0)} \\ &= 2\left[\theta + \alpha_0 \sin\theta\right]_0^{\pi - \cos^{-1}(1/\alpha_0)} - \left[\theta + \alpha_0 \sin\theta\right]_{\pi - \cos^{-1}(1/\alpha_0)}^{\pi + \cos^{-1}(1/\alpha_0)} \\ &= 3\left[\pi - \cos^{-1}(1/\alpha_0) + \alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\}\right] \\ &\quad - \left[\pi + \cos^{-1}(1/\alpha_0) - \alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\}\right] \\ &= 4\pi + \left[-2\pi - 4\cos^{-1}(1/\alpha_0) + 4\alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\}\right].\end{aligned} \tag{2.4}$$

Therefore, we will get the univalence of f in the unit disk by Ozaki–Umezawa's Lemma 1.1, whenever

$$-2\pi - 4\cos^{-1}(1/\alpha_0) + 4\alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\} = 0. \tag{2.5}$$

If $t_0 = \cos^{-1}(1/\alpha_0)$, then (2.5) becomes

$$-2\pi - 4t_0 + 4\frac{1}{\cos t_0} \sin t_0 = 0,$$

which is assumed in (2.2). Note that $1.22 < t_0 < 1.23$. □