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Partial differential equations/Probability theory

On time regularity of stochastic evolution equations with monotone coefficients

Sur la régularité en temps d'équations d'évolution stochastiques à coefficients monotones

Dominic Breit [a], Martina Hofmanová [b]

[a] *Department of Mathematics, Heriot-Watt University, Riccarton Edinburgh EH14 4AS, UK*
[b] *Technical University Berlin, Institute of Mathematics, Straße des 17. Juni 136, 10623 Berlin, Germany*

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ABSTRACT

We report on a time regularity result for stochastic evolutionary PDEs with monotone coefficients. If the diffusion coefficient is bounded in time without additional space regularity, we obtain a fractional Sobolev-type time regularity of order up to $\frac{1}{2}$ for a certain functional $G(u)$ of the solution. Namely, $G(u) = \nabla u$ in the case of the heat equation and $G(u) = |\nabla u|^{\frac{p-2}{2}} \nabla u$ for the p-Laplacian. The motivation is twofold. On the one hand, it turns out that this is the natural time regularity result that allows us to establish the optimal rates of convergence for numerical schemes based on a time discretization. On the other hand, in the linear case, i.e. when the solution is given by a stochastic convolution, our result complements the known stochastic maximal space–time regularity results for the borderline case not covered by other methods.

RÉSUMÉ

On étudie des résultats de régularité en temps pour des équations aux dérivées partielles stochastiques à coefficients monotones. Si le coefficient de diffusion est borné en temps, sans faire d'hypothèses supplémentaires sur la régularité en espace, on obtient une régularité en temps de type Sobolev fractionnaire d'ordre $\frac{1}{2}$ pour une certaine fonction $G(u)$ de la solution u. Plus précisément, $G(u) = \nabla u$ dans le cas de l'équation de la chaleur et $G(u) = |\nabla u|^{\frac{p-2}{p}} \nabla u$ pour le p-laplacien. La motivation est double : d'une part, il apparaît que ceci correspond à un résultat naturel de régularité en temps et, de plus, on obtient les taux de convergence optimaux pour les schémas de discrétisation en temps ; d'autre part, dans le cas linéaire, c'est-à-dire dans celui où la solution est donnée par une convolution stochastique, le résultat obtenu complète les résultats connus de régularité maximale dans l'espace-temps pour le cas limite, résultats qu'on ne peut pas obtenir par d'autres méthodes.

E-mail addresses: d.breit@hw.ac.uk (D. Breit), hofmanov@math.tu-berlin.de (M. Hofmanová).

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1. Time regularity

Let H, U be separable Hilbert spaces and let V be a Banach space such that $V \hookrightarrow H \hookrightarrow V'$ is a Gelfand triple with continuous and dense embeddings. We are interested in stochastic evolution equations of the form

$$\begin{aligned} \mathrm{d}u &= A(t,u)\,\mathrm{d}t + B(t,u)\,\mathrm{d}W, \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where W is a U-valued cylindrical Wiener process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a normal filtration $(\mathscr{F}_t)$ and the maps

$$A : \Omega \times [0,T] \times V \to V', \qquad B : \Omega \times [0,T] \times H \to L_2(U;H)$$

are $(\mathscr{F}_t)$-progressively measurable and satisfy

(H1) monotonicity: there exists $c_1 \in \mathbb{R}$ such that for all $u, v \in V$, $t \in [0,T]$

$$2\,{}_{V'}\langle A(t,u) - A(t,v), u - v\rangle_V + \|B(t,u) - B(t,v)\|^2_{L_2(U;H)} \le c_1 \|u - v\|^2_H;$$

(H2) hemicontinuity: for all $u, v, w \in V$, $\omega \in \Omega$ and $t \in [0,T]$, the mapping

$$\mathbb{R} \ni \lambda \mapsto {}_{V'}\langle A(\omega, t, u + \lambda v), w\rangle_V$$

is continuous;

(H3) coercivity: there exist $q \in (1,\infty)$, $c_2 \in [0,\infty)$, $c_3 \in \mathbb{R}$ such that for all $u \in V$, $t \in [0,T]$

$${}_{V'}\langle A(t,u), u\rangle_V \le -c_2 \|u\|^q_V + c_3;$$

(H4) growth of A: there exists $c_4 \in (0,\infty)$ such that for all $u \in V$, $t \in [0,T]$

$$\|A(t,u)\|^{q'}_{V'} \le c_4 \big(1 + \|u\|^q_V\big);$$

(H5) growth of B: there exists $c_5 \in (0,\infty)$ and $(\mathscr{F}_t)$-adapted $f \in L^2(\Omega; L^\infty(0,T))$ such that for all $u \in H$, $t \in [0,T]$

$$\|B(t,u)\|_{L_2(U;H)} \le c_5 (f + \|u\|_H).$$

The literature devoted to the study of these equations is quite extensive. The question of the existence of a unique (variational) solution to equations of the form (1.1) is well understood: first results were established in [15,14]; for an overview of the above-stated generality and further references, we refer the reader to [16]. The existence of a strong solution under various assumptions appeared in [3,10], and numerical approximations were studied in [11,12]. In the case of linear operator A that generates a strongly continuous semigroup, more is known concerning regularity and maximal regularity (see e.g. [6,13,17]).

Naturally, the time regularity of a solution to (1.1) is limited by the regularity of the driving Wiener process W. In particular, since the trajectories of W are only α-Hölder continuous for $\alpha < \frac{1}{2}$, it can be seen from the integral formulation of (1.1) that the trajectories of u are α-Hölder continuous as functions taking values in V'. This can be improved if some additional regularity in the space of the solution is known, that is, the equation is satisfied in a stronger sense. In this note, we are particularly interested in situations where such additional space regularity is either not available or limited. This is typically the case when

(i) A is linear but the noise is not smooth enough: if u is a variational solution to (1.1), then the standard assumption is $B(u) \in L^2_{w^*}(\Omega; L^\infty(0,T; L_2(U;H))$;[1]

(ii) A is nonlinear as for instance the p-Laplacian $A(u) = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ or a more general nonlinear operator with p-growth and, in addition, the noise represents the same difficulty as in (i).

In order to formulate our main result, we need several additional assumptions upon the operator A and the initial datum u_0. On the one hand, we introduce a notion of G-monotonicity that represents a stronger version of the monotonicity assumption on A; on the other hand, we suppose a certain regularity in time of A as well as a regularity of the initial condition. To be more precise, we assume

[1] Here $L^2_{w^*}(\Omega; L^\infty(0,T; L_2(U;H))$ is the space of weak*-measurable mappings $h : \Omega \to L^\infty(0,T; L_2(U;H))$ such that $\mathbb{E}\,\mathrm{esssup}_{0\le t\le T} \|h\|^2_{L_2(U;H)} < \infty$.