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Fixed point strategies for mixed variational formulations of the stationary Boussinesq problem ☆

Stratégies de point fixe pour formulations variationnelles mixtes du problème stationnaire de Boussinesq

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ABSTRACT

In this paper, we report on the main results concerning the solvability analysis of two new mixed variational formulations for the stationary Boussinesq problem. More precisely, we introduce mixed-primal and fully-mixed approaches, both of them suitably augmented with Galerkin-type equations, and show that the resulting schemes can be rewritten, equivalently, as fixed-point operator equations. Then, classical arguments from linear and nonlinear functional analysis are employed to conclude that they are well-posed.

RÉSUMÉ

Dans cet article, on présente les principaux résultats concernant l'analyse de résolution de deux nouvelles formulations variationnelles mixtes pour le problème stationnaire de Boussinesq. Plus précisément, on introduit des approches mixtes-primal et entièrement mixtes, toute les deux convenablement augmentées avec des équations de type Galerkin, et l'on montre que les régimes qui en résultent peuvent être réécrits, de maniére équivalente, comme équations d'opérateur de point fixe. Ainsi, les arguments classiques de l'analyse fonctionnelle linéaires et non linéaires sont utilisés pour conclure qu'elles sont bien posées.

1. Introduction

We first let $\Omega \subseteq \mathrm{R}^n$, $n \in \{2, 3\}$, be a given bounded domain with polyhedral boundary Γ, denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ, and consider a fluid occupying Ω. Throughout this work, a standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote

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the corresponding vectorial and tensorial counterparts of the generic scalar functional space M. Then, given a fluid viscosity $\mu > 0$, an external force per unit mass $\boldsymbol{g} \in \mathbf{L}^\infty(\Omega)$, a boundary velocity $\boldsymbol{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, a boundary temperature $\varphi_D \in \mathrm{H}^{1/2}(\Gamma)$, and a uniformly positive definite tensor $\mathbb{K} \in \mathbb{L}^\infty(\Omega)$ describing the thermal conductivity, the stationary Boussinesq problem reads: find the velocity $\boldsymbol{u}$, the pressure p, and the temperature φ of the fluid such that

$$\begin{aligned} -\mu\,\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\,\boldsymbol{u} + \nabla p - \boldsymbol{g}\,\varphi \;=\; 0 \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} \;=\; 0 \quad &\text{in} \quad \Omega\,, \\ -\mathrm{div}(\mathbb{K}\,\nabla\varphi) + \boldsymbol{u}\cdot\nabla\varphi \;=\; 0 \quad &\text{in} \quad \Omega\,, \\ \boldsymbol{u} \;=\; \boldsymbol{u}_D \quad \text{and} \quad \varphi \;=\; \varphi_D \quad &\text{on} \quad \Gamma\,. \end{aligned} \tag{1}$$

Note that $\boldsymbol{u}_D$ must satisfy the compatibility condition $\int_\Gamma \boldsymbol{u}_D\cdot\boldsymbol{\nu} = 0$, which comes from the incompressibility condition of the fluid. In turn, we also notice that the uniqueness of a pressure solution to (1) (see, e.g., [8]), is ensured in the space $\mathrm{L}^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}$. Next, introducing the auxiliary tensor unknown $\boldsymbol{\sigma} := \mu\,\nabla\boldsymbol{u} - (\boldsymbol{u}\otimes\boldsymbol{u}) - p\,\mathbb{I}$ in Ω, where $\mathbb{I}$ is the identity matrix of $\mathrm{R}^{n\times n}$, using the incompressibility condition to eliminate the pressure unknown by means of the formula $p = -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma} + \boldsymbol{u}\otimes\boldsymbol{u})$ in Ω, and denoting the deviatoric of a tensor $\boldsymbol{\tau}$ by $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}$, we arrive at the following system of equations with unknowns $\boldsymbol{u}$, $\boldsymbol{\sigma}$, and φ

$$\begin{aligned} \mu\,\nabla\boldsymbol{u} - (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} \;=\; \boldsymbol{\sigma}^{\mathrm{d}} \quad \text{and} \quad -\,\mathbf{div}\,\boldsymbol{\sigma} - \boldsymbol{g}\,\varphi \;=\; 0 \quad &\text{in} \quad \Omega\,, \\ -\mathrm{div}(\mathbb{K}\,\nabla\varphi) + \boldsymbol{u}\cdot\nabla\varphi \;=\; 0 \quad &\text{in} \quad \Omega\,, \\ \boldsymbol{u} \;=\; \boldsymbol{u}_D \quad \text{and} \quad \varphi \;=\; \varphi_D \quad &\text{on} \quad \Gamma\,, \\ \int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \boldsymbol{u}\otimes\boldsymbol{u}) \;=\; 0\,. \quad & \end{aligned} \tag{2}$$

In the following sections we propose and analyze two new augmented mixed variational formulations for (2). For other approaches concerning this and related problems, we refer to [1,5,7,8], and the references therein.

2. The augmented mixed-primal formulation

In this section we consider an augmented mixed approach for the equations modeling $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, whereas a primal formulation is employed to deal with the main equation modeling the temperature φ.

2.1. The continuous formulation

In what follows we make use of the decomposition (see e.g. [2,6]) $\mathbb{H}(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div};\Omega) \oplus \mathrm{R}\,\mathbb{I}$, where

$$\mathbb{H}(\mathbf{div};\Omega) := \left\{ \boldsymbol{\zeta} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\zeta} \in \mathbf{L}^2(\Omega) \right\}, \quad \text{and} \quad \mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) = 0 \right\}.$$

In particular, $\boldsymbol{\sigma}$ in (2) can be decomposed as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbb{I}$, where $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$, and, thanks to the last equation in (2), c is given explicitly in terms of $\boldsymbol{u}$ as $c = -\frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{u}\otimes\boldsymbol{u})$. Then, renaming $\boldsymbol{\sigma}_0$ as $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$, noting that the first and second equations of (2) remain unchanged, multiplying all the equations of (2), except the last one, by suitable test functions, integrating by parts whenever it is necessary, incorporating the Dirichlet condition for $\boldsymbol{u}$ (which is a natural boundary condition in this case), introducing $\lambda := -\mathbb{K}\,\nabla\varphi\cdot\boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\Gamma)$ as a new unknown, imposing the Dirichlet condition for φ weakly, and denoting by $\langle\cdot,\cdot\rangle_\Gamma$ the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$) and $\mathrm{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$), we first obtain the following set of equations:

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \mu\int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \;=\; \mu\,\langle\boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{u}_D\rangle_\Gamma \qquad &\forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)\,, \\ -\mu\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \mu\int_\Omega \varphi\,\boldsymbol{g}\cdot\boldsymbol{v} \;=\; 0 \qquad &\forall\,\boldsymbol{v} \in \mathbf{L}^2(\Omega)\,, \\ \int_\Omega \mathbb{K}\,\nabla\varphi\cdot\nabla\psi + \langle\lambda,\psi\rangle_\Gamma + \int_\Omega (\boldsymbol{u}\cdot\nabla\varphi)\,\psi \;=\; 0 \qquad &\forall\,\psi \in \mathrm{H}^1(\Omega)\,, \\ \langle\xi,\varphi\rangle_\Gamma \;=\; \langle\xi,\varphi_D\rangle_\Gamma \qquad &\forall\,\xi \in \mathrm{H}^{-1/2}(\Gamma)\,. \end{aligned} \tag{3}$$