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Classification of plethories in characteristic zero

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ABSTRACT

We classify plethories over fields of characteristic zero, thus answering a question of Borger–Wieland and Bergman. All plethories over characteristic zero fields are linear, in the sense that they are free plethories on a bialgebra. For the proof we need some facts from the theory of ring schemes where we extend previously known results. We also classify plethories with trivial Verschiebung over a perfect field of non-zero characteristic and indicate future work.

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1. Introduction

Plethories, first introduced by Tall–Wraith [13], and then studied by Borger–Wieland [3], are precisely the objects which act on k-algebras, for k a commutative ring. There are

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many fundamental questions regarding plethories which remain unanswered. One such question is, given a ring k, whether one can classify plethories over k, in this paper we will take a first step towards a classification.

For some motivation, let us start by looking at the category of modules $\mathrm{Mod_k}$ over a commutative ring k. If we consider the category of representable functors $\mathrm{Mod_k} \to \mathrm{Mod_k}$, there is a monoidal structure given by composition of functors. Then one defines a k-algebra R as a k-module R together with a comonad structure on the representable endofunctor $\mathrm{Mod_k}(R, -) : \mathrm{Mod_k} \to \mathrm{Mod_k}$ with respect to composition of functors. Heuristically, this says that a k-algebra is precisely the kind of object which knows how to act on k-modules. This can be extended to a non-linear setting, so that instead of looking at k-modules we look at k-algebras $\mathrm{Alg_k}$ and consider representable endofunctors $\mathrm{Alg_k} \to \mathrm{Alg_k}$. A comonoid with respect to composition of functors is then called a plethory and analogously, a plethory is what knows how to act on k-algebras. One particular important example of a plethory is the $\mathbb{Z}$-algebra Λ which consists of the ring of symmetric functions in infinitely many variables with a certain biring structure. The functor $\mathrm{Alg_k}(\Lambda, -) : \mathrm{Alg_k} \to \mathrm{Alg_k}$ represents the functor taking a ring R to its ring of Witt vectors. Using plethories one gets a very conceptual view of Witt vectors and in [2] James Borger develops the geometry of Witt vectors using the plethystic perspective.

Let now k be a field. If we let $\mathcal{P}_k$ denote the category of plethories over k, there is a functor

$$P : \mathcal{P}_k \to \mathrm{Bialg_k},$$

into the category of cocommutative counital bialgebras over k, which takes a plethory Q to its primitive elements $P(Q)$. This functor has a left adjoint $\mathrm{S}(-) : \mathrm{Bialg_k} \to \mathcal{P}$ and we say that a plethory Q is linear if $Q \cong \mathrm{S}(A)$ for some cocommutative, counital bialgebra A. Heuristically, a plethory Q is linear if every action of Q on an algebra B comes from an action of a bialgebra on B. The main theorem of this paper is:

Theorem 1.1. *Let k be a field of characteristic zero. Then any k-plethory is linear.*

This answers a question of Bergman–Hausknecht [1, p. 336] and Borger–Wieland [3] in the positive. The theorem is proved by studying the category of affine ring schemes. We have the following results, extending those of Greenberg [8] to arbitrary fields and not necessarily reduced schemes:

Theorem 1.2. *Let k be a field. Then any connected ring scheme of finite type is unipotent.*

Theorem 1.3. *Let P be a connected ring scheme of finite type over k. Then P is affine.*