Journal of Algebra 516 (2018) 210–244

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Structural decomposition of monomial resolutions

Guillermo Alesandroni

Department of Mathematics, Wake Forest University, 1834 Wake Forest Rd, Winston-Salem, NC 27109, United States of America

A R T I C L E   I N F O

Article history:
Received 8 June 2017
Available online 11 September 2018
Communicated by Luchezar L. Avramov

Keywords:
Taylor resolution
Monomial ideal
Minimal free resolution
Projective dimension
Multigraded Betti number

A B S T R A C T

We express the multigraded Betti numbers of an arbitrary monomial ideal in terms of the multigraded Betti numbers of two basic classes of ideals. This decomposition has multiple applications. In some concrete cases, we use it to construct minimal resolutions of classes of monomial ideals; in other cases, we use it to compute projective dimensions. To illustrate the effectiveness of the structural decomposition, we give a new proof of a classic theorem by Charalambous that states the following: let k be a field, and M an Artinian monomial ideal in $S = k[x_1, \ldots, x_n]$; then, for all i, $\mathrm{b}_i(S/M) \geq \binom{n}{i}$.

1. Introduction

The problem of finding the minimal resolution of an arbitrary monomial ideal in closed form has been deemed utopic by many a mathematician. As a consequence, people have tried to restrict the study of minimal resolutions to particular classes of ideals. Borel ideals, minimally resolved by the Eliahou–Kervaire resolution [6]; generic ideals, minimally resolved by the Scarf complex [2]; and dominant ideals, minimally resolved by the Taylor resolution [1], are examples of this restrictive approach.

E-mail address: alesangc@wfu.edu.

https://doi.org/10.1016/j.jalgebra.2018.09.003
0021-8693/

In the first half of this paper, however, we turn to the general problem, and decompose the minimal resolution of an arbitrary monomial ideal in terms of the minimal resolutions of two basic classes that we call dominant, and purely nondominant ideals. More precisely, we express the multigraded Betti numbers of an ideal as the sum of the multigraded Betti numbers of some dominant and some purely nondominant ideals. Since dominant ideals are minimally resolved by their Taylor resolutions, our decomposition reduces the study of minimal monomial resolutions to the study of minimal resolutions of purely nondominant ideals.

Unfortunately, the resolutions of purely nondominant ideals involve the same challenges that we encounter in the general context. Some of these difficulties are the existence of ghost terms (that is, the minimal resolution contains basis elements of equal multidegree in consecutive homological degrees), characteristic dependence, and the striking fact that some of the simplest purely nondominant ideals cannot be minimally resolved by any subcomplex of the Taylor resolution. Thus, in the second half of this work we focus our efforts on one particular case: monomial ideals whose structural decomposition has no purely nondominant part. As a result of this study, we obtain the multigraded Betti numbers of two families that we call 2-semidominant and almost generic ideals.

The structural decomposition is also a useful tool to compute projective dimensions. We prove, for instance, that if an ideal M satisfies certain conditions, $\mathrm{pd}(S/M) = 2$, and, under some other conditions, $\mathrm{pd}(S/M) = n$, where n is the number of variables in the polynomial ring. Another result, also related to projective dimensions, is a new proof of a classic theorem of Charalambous [3] (see also [12, Corollary 21.6]), stating: let k be a field, and M an Artinian monomial ideal in $S = k[x_1, \ldots, x_n]$; then, for all i, $\mathrm{b}_i(S/M) \geq \binom{n}{i}$. While the original proof relies on the radical of an ideal, ours is based on the structural decomposition.

The organization of the article is as follows. Section 2 is about background and notation. Section 3 is technical; it contains some isomorphism theorems that will be used to construct the structural decomposition of section 4. In sections 5 and 6 we compute Betti numbers and projective dimensions using the structural decomposition. Section 7 relates the structural decomposition to the lcm-lattice, and describes connections to some classes of ideals combinatorially defined. Section 8 is the conclusion; it includes some comments, questions, and conjectures.

2. Background and notation

Throughout this paper S represents a polynomial ring over an arbitrary field k, in a finite number variables. The letter M always denotes a monomial ideal in S. With minor modifications, the constructions that we give below can be found in [11,12].

Construction 2.1. Let M be generated by a set of monomials $\{l_1, \ldots, l_q\}$. For every subset $\{l_{i_1}, \ldots, l_{i_s}\}$ of $\{l_1, \ldots, l_q\}$, with $1 \leq i_1 < \ldots < i_s \leq q$, we create a formal symbol $[l_{i_1}, \ldots, l_{i_s}]$, called a **Taylor symbol**. The Taylor symbol associated to $\{\}$ is denoted by $[\varnothing]$.