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# On sums of two and three roots of unity

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A B S T R A C T

Let $\alpha \neq 0$ be a sum of some $k$ distinct $N$th roots of unity, where $2 \leq k < N$. In 1986, Myerson raised the following two problems. How small can $|\alpha|$ be? How large can the modulus of the product of all conjugates of $\alpha$ lying in the disc $|z| < 1$ be? A simple Liouville type argument gives the lower bound $k^{-N+2}$ for these quantities, so the problem is to find appropriate upper bounds. As for the first question, for $k \geq 5$, it remains a huge gap between lower and the best known upper bound $N^{-d_k}$. In this note, we give a complete answer to the second question of Myerson for $k = 2$. For $k = 3$ and $N$ large prime, we show that a positive proportion of the conjugates of any such $\alpha$ lie in the disc $|z| \leq \varrho$, where $\varrho < 1$. This implies a naturally expected upper bound.



## 1. Introduction

Throughout, for an algebraic integer $\alpha \neq 0$ of degree $D$ over $\mathbb{Q}$ with conjugates $\alpha_1, \dots, \alpha_D$, we denote by $\Lambda(\alpha)$ the modulus of the product of the conjugates of $\alpha$ lying inside the unit circle:

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$$\Lambda(\alpha) := \prod_{j=1}^{D} \min\{1, |\alpha_j|\}.$$

Clearly,

$$\Lambda(\alpha) = \frac{|\mathrm{Norm}(\alpha)|}{M(\alpha)}, \tag{1}$$

where $M(\alpha) := \prod_{j=1}^{D} \max\{1, |\alpha_j|\}$ is the Mahler measure of $\alpha$ and $\mathrm{Norm}(\alpha) := \prod_{j=1}^{D} \alpha_j$ is the norm of $\alpha$.

Let $k$ and $N$ be two positive integers satisfying $2 \le k < N$. In [12], Myerson considered the set $A(k, N)$ of algebraic numbers $\alpha$ that can be expressed by the sum of $k$ distinct $N$th roots of unity, that is,

$$A(k, N) := \{\zeta^{a_1} + \cdots + \zeta^{a_k}\},$$

where $\zeta = e^{2\pi i/N}$ and $a_1, \dots, a_k \in \mathbb{Z}$ satisfy $0 \le a_1 < \cdots < a_k \le N - 1$.

Since each $\alpha \in A(k, N) \setminus \{0\}$ belongs to the field $\mathbb{Q}(e^{2\pi i/N})$, its degree $\deg \alpha$ divides $\varphi(N)$, where $\varphi$ is Euler's totient function. In fact, the union of all such sets $A(k, N)$ represents the set of algebraic integers of the maximal abelian extension of $\mathbb{Q}$. See, e.g., Lemma 5 in [1] for an upper bound of the cardinality of such numbers.

In [12], motivated by [11] and assuming that $k$ is fixed and $N$ is large, Myerson raised the following two problems for $\alpha \in A(k, N) \setminus \{0\}$:

- How small in terms of $k$ and $N$ (or $\deg \alpha$) can $|\alpha|$ be?
- How large in terms of $k$ and $N$ (or $\deg \alpha$) can $\Lambda(\alpha) < 1$ be?

One can easily give some lower bounds using a simple Liouville type argument. Let $t \ge 1$ be the number of conjugates of $\alpha \in A(k, N) \setminus \{0\}$, lying in $|z| < 1$. (Evidently, $\Lambda(\alpha) = 1$ if $t = 0$.) Since $|\mathrm{Norm}(\alpha)| \ge 1$ and each conjugate of such $\alpha$ has modulus at most $k$, by (1), we obtain

$$\Lambda(\alpha) \ge M(\alpha)^{-1} \ge k^{-\deg \alpha + t} \ge k^{-\varphi(N)+t} \ge k^{-N+2}. \tag{2}$$

Hence,

$$|\alpha| \ge \Lambda(\alpha) \ge k^{-N+2} \tag{3}$$

for each nonzero $\alpha \in A(k, N)$.

As for the bound (3), one should say that the best upper bounds obtained for some $\alpha \in A(k, N)$ are of the form

$$|\alpha| \ll N^{-d_k}, \tag{4}$$

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