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Some results for the irreducibility of truncated binomial expansions ☆

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For positive integers k and n with $k \leqslant n-1$, let $P_{n,k}(x)$ denote the polynomial $\sum_{j=0}^{k} \binom{n}{j} x^j$, where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$. In 2011, Khanduja, Khassa and Laishram proved the irreducibility of $P_{n,k}(x)$ over the field $\mathbb{Q}$ of rational numbers for those n, k for which $2 \leq 2k \leq n < (k+1)^3$. In this paper, we extend the above result and prove that if $2 \leq 2k \leq n < (k+1)^{e+1}$ for some positive integer e and the smallest prime factor of k is greater than e, then there exists an explicitly constructible constant C_e depending only on e such that the polynomial $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ for $k \geq C_e$.

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1. Introduction

For positive integers k and n with $k \leqslant n-1$, let $P_{n,k}(x)$ denote the polynomial $\sum_{j=0}^{k}\binom{n}{j}x^j$, where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$. In 2007, Filaseta, Kumchev and Pasechnik [2] considered the problem of irreducibility of $P_{n,k}(x)$ over the field $\mathbb{Q}$ of rational numbers. They proved that for any fixed integer $k \geqslant 3$,[1] there exists an integer n_0 depending on k such that $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ for every $n \geqslant n_0$. In 2011, Khanduja, Khassa and Laishram proved the irreducibility of $P_{n,k}(x)$ for those n, k for which $2 \leq 2k \leqslant n < (k+1)^3$ (cf. [3]). In this paper we extend the above result and prove the following theorem.

Theorem 1.1. *Let k, n, e be positive integers with $2 \leq 2k \leqslant n < (k+1)^{e+1}$. Let M_e denote the integer $\frac{(e+1)(3e+1)}{4}$ if e is odd and $\frac{e(3e+2)}{4}$ if e is an even integer. Let L_e denote the smallest integer greater than or equal to $\frac{4}{3}(M_e+2)$. If k is greater than or equal to the ${L_e}^{th}$ prime number, then either $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ or it has a factor of degree $\frac{ik}{j}(\leq \frac{k}{2})$ for some $1 \leq i \leq \left[\dfrac{e+1}{2}\right]$, $j \leq e$, where $[r]$ stands for the greatest integer not exceeding r.*

It may be pointed out that when $e = 2$ in the above theorem, then either $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ or it has a factor of degree $\frac{k}{2}$ for $38 \leq 2k \leq n < (k+1)^3$. Also when $106 \leq 2k \leq n < (k+1)^4$ and $P_{n,k}(x)$ does not have factors of degree $\frac{k}{3}$, $\frac{k}{2}$, then $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$.

We indeed prove the following slightly stronger result from which Theorem 1.1 quickly follows.

Theorem 1.2. *Let k, n be positive integers such that $2k \leqslant n$. Let e be the maximum positive integer such that there exists[2] a prime $p > k$ dividing $n(n-1)\cdots(n-k+1)$ with exact power e. Let M_e, L_e be as in Theorem 1.1 and p_{L_e} denote the ${L_e}^{th}$ prime number. If $k \geq p_{L_e}$, then either $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ or it must have a factor of degree $\frac{ik}{j}(\leq \frac{k}{2})$ for some $1 \leq i \leq \left[\dfrac{e+1}{2}\right]$ with $j \leq e$.*

The following corollary yields irreducibility of truncated binomial for certain n, k and immediate consequence of the above theorem.

[1] For $k = 2$, $P_{n,k}(x)$ has negative discriminant and hence is irreducible over $\mathbb{Q}$.

[2] Sylvester [5] proved in 1892 that a product of k consecutive numbers $n, n-1, \cdots, n-k+1$ with $n \geq 2k$ is divisible by a prime exceeding k.

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