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On unbounded denominators and hypergeometric series

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ABSTRACT

We study the question of when the coefficients of a hypergeometric series are p-adically unbounded for a given rational prime p. Our first result is a necessary and sufficient criterion, applicable to all but finitely many primes, for determining when the coefficients of a hypergeometric series with rational parameters are p-adically unbounded (equivalent but different conditions were found earlier by Dwork in [12] and Christol in [9]). We then show that the set of unbounded primes for a given series is, up to a finite discrepancy, a finite union of the set of primes in certain arithmetic progressions and we explain how this set can be computed. We characterize when the density of the set of unbounded primes is 0 (a similar result is found in [9]), and when it is 1. Finally, we discuss the connection between this work and the unbounded denominators conjecture for Fourier coefficients of modular forms.

Contents

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1. Introduction

Hypergeometric series are objects of considerable interest. From a number theoretic perspective, hypergeometric differential equations provide a convenient and explicit launching point for subjects such as p-adic differential equations [13], [14], rigid differential equations and Grothendieck's p-curvature conjecture [22], as well as the study of periods and motives. Recent attention has focused on the relationships between quotient singularities, integer ratios of factorials, and the Riemann hypothesis [7], [28], [6]. It is the Beukers–Heckman [5] classification of generalized hypergeometric differential equations with finite monodromy that underlies these connections.

If $F(z)$ is a solution of an ordinary differential equation (Fuchsian on $\mathbb{P}^1$, say) with a finite monodromy group, then it is an algebraic function. Moreover if $F(z)$ has rational Taylor coefficients, then an old theorem of Eisenstein states that for some integer N, the series $F(Nz)$ has integer coefficients, save for possibly the constant term (see [15] for an interesting discussion of this result). This says two things:

1. $F(z)$ has p-adically bounded coefficients for almost all primes p;
2. for those primes for which $F(z)$ has p-adically unbounded coefficients, the coefficients cannot grow too quickly in p-adic absolute value.

(We say that $F(z)$ has p-adically unbounded coefficients when arbitrarily high powers of p appear in the denominators of the coefficients.) In the present paper, given a hypergeometric series with rational coefficients, we study the set of all primes for which the series has p-adically unbounded coefficients. We do not assume that the monodromy is finite, although we do impose some mild restrictions, such as irreducibility of the monodromy representation. The discussion at the start of Section 3 gives a precise description of the conditions that we impose. See [11], [12], [9] and [10] for earlier work on this subject. The work [10] and the references contained therein describe how the subject of p-integrality of hypergeometric series plays a role in the study of so-called mirror maps arising in mathematical physics. See also [1] for a discussion of p-integrality of A-hypergeometric series, and [21] for a related discussion of hypergeometric series with parameters taken in a quadratic extension of the rational numbers.

The basic tool that we use to study hypergeometric series is an old result of Kummer (cf. Theorem 2.5), characterizing the p-adic valuation of binomial coefficients in terms of counting p-adic carries in certain p-adic additions. In Theorem 3.4 we employ Kummer's result to deduce a formula for the p-adic valuation of the coefficients of a generalized

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