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On the smallest number of terms of vanishing sums of units in number fields ☆

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ABSTRACT

Let K be a number field. In the terminology of Nagell a unit ε of K is called *exceptional* if $1-\varepsilon$ is also a unit. The existence of such a unit is equivalent to the fact that the unit equation $\varepsilon_1+\varepsilon_2+\varepsilon_3=0$ is solvable in units $\varepsilon_1,\varepsilon_2,\varepsilon_3$ of K. Numerous number fields have exceptional units. They have been investigated by many authors, and they have important applications.
In this paper we deal with a generalization of exceptional units. We are interested in the smallest integer k with $k\geq 3$, denoted by $\ell(K)$, such that the unit equation $\varepsilon_1+\cdots+\varepsilon_k=0$ is solvable in units $\varepsilon_1,\dots,\varepsilon_k$ of K. If no such k exists, we set $\ell(K)=\infty$. Apart from trivial cases when $\ell(K)=\infty$, we give an explicit upper bound for $\ell(K)$. We obtain several results for $\ell(K)$ in number fields of degree at most 4, cyclotomic fields and general number fields of given degree. We prove various properties of $\ell(K)$, including its magnitude, parity as well as

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the cardinality of number fields K with given degree and given odd resp. even value $\ell(K)$.
Finally, as an application, we deal with certain arithmetic graphs, namely we consider the representability of cycles. We conclude the paper by listing some problems and open questions.

1. Introduction

Let K be a number field. We are interested in the smallest integer k having the following property:

$$\text{there exist units } \varepsilon_1, \ldots, \varepsilon_k \in K \text{ such that } \varepsilon_1 + \cdots + \varepsilon_k = 0. \tag{1}$$

Observe that for any even integer $k = 2t$, we have a trivial assertion given by $t \times 1 + t \times (-1) = 0$. So we shall use the following definitions.

Write $\ell_o(K)$ for the smallest odd $k \geq 3$ for which (1) is valid. Further, let $\ell_e(K)$ be the smallest even $k \geq 4$ for which (1) is valid, such that the sum appearing in (1) has no proper vanishing subsum. If no appropriate k exists at all, then we set $\ell_o(K) = \infty$ or $\ell_e(K) = \infty$, respectively. We put $\ell(K) = \min(\ell_o(K), \ell_e(K))$.

Before proceeding further, we make a trivial observation. Namely, if $k = \ell_o(K)$, then the sum of units appearing in (1) has no proper vanishing subsum. Indeed, otherwise we would have a proper vanishing subsum with an odd number of terms, contradicting the minimality of $k = \ell_o(K)$.

The above notions can be generalized to orders of number fields. Let $\mathcal{O}$ be an order of a number field K. Then we can define $\ell_o(\mathcal{O}), \ell_e(\mathcal{O}), \ell(\mathcal{O})$ in the obvious way. Note that if $\mathcal{O}$ is the maximal order of K, then we clearly have $\ell_o(\mathcal{O}) = \ell_o(K)$, $\ell_e(\mathcal{O}) = \ell_e(K)$, $\ell(\mathcal{O}) = \ell(K)$.

In this paper we obtain several results concerning $\ell(K), \ell_o(K), \ell_e(K)$ and $\ell(\mathcal{O}), \ell_o(\mathcal{O}), \ell_e(\mathcal{O})$. We show among other things that $\ell(K)$ is finite for any number field K, apart from the cases where $K = \mathbb{Q}$ or K is an imaginary quadratic field. Further, we prove that for any integer $k \geq 3$ there exists an order of a real quadratic number field with $\ell(\mathcal{O}) = k$, and also a complex cubic number field K with $\ell(K) = k$ – in the latter case excluding values k of the form $k = 4t^4 - 4t + 2$. On the other hand, we show that for each k, there are only finitely many quadratic fields, complex cubic fields and (up to certain completely described exceptions) totally complex quartic fields with $\ell(K) \leq k$, and all these number fields can be effectively determined. Furthermore, it is shown that for any number field K different from $\mathbb{Q}$ and the imaginary quadratic fields we have $\ell_e(K) < \infty$. Finally, we prove that for $d \geq 3$ there are infinitely many number fields K of degree d with $\ell_e(K) = 4$, and for $d \geq 2$ there are infinitely many number fields K of degree d with $\ell_o(K) = \infty$.

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