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Mock characters and the Kronecker symbol ☆

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ABSTRACT

We introduce and study a family of functions we call the *mock characters*. These functions satisfy a number of interesting properties, and of all completely multiplicative arithmetic functions seem to come as close as possible to being Dirichlet characters. Along the way we prove a few new results concerning the behaviour of the Kronecker symbol.

One of the most familiar objects in number theory is the Kronecker symbol, denoted $\left(\frac{a}{n}\right)$ or $(a|n)$. Viewed as a function of n, it is well-known that this is a primitive real character when a is a fundamental discriminant. Less well-known is the behavior when a is *not* a fundamental discriminant; in this case $\left(\frac{a}{\cdot}\right)$ might be a primitive character (e.g., for $a = 2$), an imprimitive character (e.g., for $a = 4$), or not a character at all

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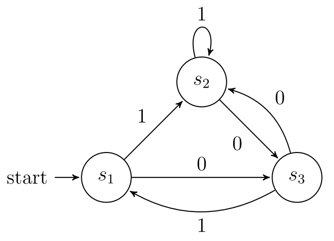

Fig. 1. For this FSM, the binary program 1101 yields the output s_2.

(e.g., for $a = 3$).[1] Even when $(\frac{a}{\cdot})$ is not a character, it strongly mimics the behavior of one, replacing the condition of periodicity with *automaticity* (a notion we shall discuss below). Inspired by this example, we define and study a family of character-like functions which we call *mock characters*. Of all completely multiplicative arithmetic functions, the mock characters are as close as possible to being characters. We will justify this statement qualitatively, and also formulate a quantitative conjecture using the language of 'pretentiousness' introduced by Granville and Soundararajan in [29].

The structure of the paper is as follows. In Section 1 we briefly review automatic sequences. In Section 2 we introduce mock characters and prove a few basic properties. In Section 3 we explore the relationship between mock characters and the Kronecker symbol, and use our results to obtain some new results about the Kronecker symbol. In the final section, we view mock characters through the lens of the 'pretentious' approach to number theory.

1. A quick overview of automatic sequences

The goal of this section is to recall and motivate the notion of an automatic sequence. We will first give a computer-science definition, then discuss a mathematical motivation for studying automatic sequences, and finally give an equivalent definition which is easier to compute with.

Recall that a *Finite State Machine* is a finite collection of states along with transition rules between states. A positive integer corresponds to a program: its digits (read right to left) dictate the individual state transitions. [See an example on Fig. 1.]

Definition 1.1. A sequence (a_n) is called *q-automatic* if and only if there exists a Finite State Machine which outputs a_n when given the program n (written base q).

Example 1.2. A nice example of a 2-automatic sequence is the *regular paperfolding sequence*. To generate this sequence, start with a large piece of paper, and label the top left corner L and the top right corner R_0. Fold the paper in half so that R_0 ends up underneath L, and label the new upper right corner by R_1 (the paper should be oriented

[1] See Section 3.2 below, in particular, Corollary 3.3.

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