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Yan Li, Daeyeoul Kim

PII: S0022-314X(18)30142-2
DOI: https://doi.org/10.1016/j.jnt.2018.04.023
Reference: YJNTH 6042

To appear in: *Journal of Number Theory*

Received date: 9 December 2017
Revised date: 19 April 2018
Accepted date: 22 April 2018

Please cite this article in press as: Y. Li, D. Kim, Menon-type identities with additive characters, *J. Number Theory* (2018), https://doi.org/10.1016/j.jnt.2018.04.023

MENON-TYPE IDENTITIES WITH ADDITIVE CHARACTERS

YAN LI AND DAEYEOUL KIM*

ABSTRACT. The classical Menon's identity [7] states that

$$\sum_{\substack{a=1\\ \gcd(a,n)=1}}^{n} \gcd(a-1,n) = \varphi(n)\tau(n),$$

for every positive integer n, where $\varphi(n)$ is the Euler's totient function and $\tau(n)$ is the number of positive divisors of n. Recently, Zhao and Cao [19] extended Menon's identity to Dirichlet characters :

$$\sum_{\substack{a=1\\ \gcd(a,n)=1}}^{n} \gcd(a-1,n)\chi(a) = \varphi(n)\tau\left(\frac{n}{d}\right),$$

where χ is a Dirichlet character modulo n and d is the conductor of χ. In this paper, we extend Menon's identity to additive characters. A special case of our main result reads like this :

$$\sum_{\substack{a=1\\ \gcd(a,n)=1}}^{n} \gcd(a-1,n)\exp\left(\frac{ka2\pi i}{n}\right) = \exp\left(\frac{k2\pi i}{n}\right)\tau(\gcd(k,n))\varphi(n)$$

if $\mathrm{ord}_p(n) - \mathrm{ord}_p(k) \neq 1$ holds for any prime p dividing n, where for $u \in \mathbb{Z}$, $\mathrm{ord}_p(u)$ is the exponent of the highest power of p dividing u. For $k = 0$, our result reduces to the classical Menon's identity.

1. Introduction

Throughout this paper $\mathbb{N}$ and $\mathbb{Z}$ will denote sets of positive integers and integers, respectively. In 1965, P. K. Menon [7] proved the following elegant identity

$$\sum_{\substack{a=1\\ \gcd(a,n)=1}}^{n} \gcd(a-1,n) = \varphi(n)\tau(n), \tag{1}$$

valid for every positive integer n, where gcd(,) represents the greatest common divisor, φ is the Euler's totient function and $\tau(n)$ is the number of positive divisors of n.

Many authors gave various generalizations of Menon's identity in different directions. For example, Sury [12] obtained the following Menon-type identity

$$\sum_{\substack{1\le a_1,\ldots,a_s\le n\\ \gcd(a_1,n)=1}} \gcd(a_1-1,a_2,\ldots,a_s,n) = \varphi(n)\sigma_{s-1}(n), \tag{2}$$

2010 *Mathematics Subject Classification.* 11A07, 11A25.
Key words and phrases. Menon's identity, additive character, Ramanujan sum, arithmetical sum, greatest common divisor, Euler's totient function, congruence.

*Corresponding author.