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ANIHPC:2884

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Ann. I. H. Poincaré – AN ••• (••••) •••–•••

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Regularity for nonlinear stochastic games ☆

Hannes Luiro, Mikko Parviainen *

Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, FI-40014 Jyväskylä, Finland

Received 13 February 2017; received in revised form 23 October 2017; accepted 13 November 2017

Abstract

We establish regularity for functions satisfying a dynamic programming equation, which may arise for example from stochastic games or discretization schemes. Our results can also be utilized in obtaining regularity and existence results for the corresponding partial differential equations.

MSC: 91A15; 35J92; 35B65; 35J60; 49N60

Keywords: Dynamic programming principle; p-Laplace; Tug-of-war; Tug-of-war with noise with space dependent probabilities; Viscosity solutions

1. Introduction

In [12], we studied regularity for the stochastic game called tug-of-war with noise. Recalling the recently discovered connection to the p-harmonic functions [18], our results implied local Lipschitz regularity for the solutions to the p-Laplace equation for $2 < p < \infty$. The approach was based on a choice of strategies for the players, and is thus quite different from the PDE proofs.

Our argument utilized symmetry properties of strategies, and a sharp cancellation effect produced by this symmetry, which directly allowed us to obtain a local Lipschitz estimate. It is a nontrivial task to extend this method to a more general class of problems where the perfect symmetry breaks down. Thus, in this paper, we develop a more flexible regularity method. As a starting point, we take a dynamic programming equation

$$u(x) = \frac{1}{2} \sup_{\mu_1 \in \mathcal{A}_1(x)} \int_{\mathbb{R}^n} u(y)\, d\mu_1(y) + \frac{1}{2} \inf_{\mu_2 \in \mathcal{A}_2(x)} \int_{\mathbb{R}^n} u(y)\, d\mu_2(y)$$

☆ Both authors have been supported by the Academy of Finland (#260791, #292797).

* Corresponding author.
E-mail addresses: hannes.s.luiro@jyu.fi (H. Luiro), mikko.j.parviainen@jyu.fi (M. Parviainen).

https://doi.org/10.1016/j.anihpc.2017.11.009

Please cite this article in press as: H. Luiro, M. Parviainen, Regularity for nonlinear stochastic games, Ann. I. H. Poincaré – AN (2018), https://doi.org/10.1016/j.anihpc.2017.11.009

as explained in detail in Section 2.1. This is a rather general formulation that covers a wide class of games; in addition to stochastic games it may as well arise from discretization schemes in numerical methods for partial differential equations, see for example [17].

To illustrate our approach, we prove asymptotic Hölder continuity, Theorem 2.1, for several examples using the method. For expository purposes we start with the examples of the tug-of-war (Section 3) and the random walk (Section 4). In addition to symmetry issues mentioned above, a limitation in some known approaches is the requirement of translation invariance. Therefore our next example is the tug-of-war with noise with spatially dependent probabilities in Section 5. Finally, we deal with the tug-of-war with noise related to the p-Laplacian with the full range $1 < p \leq \infty$, including $1 < p < 2$, in Section 6. Proving a local regularity result for the game of Section 6 may seem difficult since the players can affect the direction of the noise, and thus it is not easy to say much about the noise distribution. However, the method of this paper is well suited for the task. We did not exhaust the list of possible examples that can be treated by the method but expect it to be useful in many more problems. Also, at least in some cases, the method can be improved to give directly stronger regularity results.

Our method arises from stochastic game theory even if for expository reasons we have eventually avoided stochastic arguments. The idea is that we start the game simultaneously at two points $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$, and try to pull the points 'closer' to each other. Here closer means, at least roughly, in the sense of averages and in terms of a suitable comparison function. To show that we may pull the points closer in this sense, we may consider the process in the higher dimensional space by setting $(x, z) \in \mathbb{R}^{2n}$, and use the subspace

$$T := \{(x, z) \in \mathbb{R}^{2n} : x = z\} \subset \mathbb{R}^{2n}$$

as a target. We use the following strategy: if our opponent takes a non optimal step we pull directly towards T. If the opponent pulls (almost) away from T, then our strategy is to aim at the exactly opposite step. The curvature of the comparison function gives an advantage to us. It is also worth noting that there is a freedom to choose among the probability measures in $\mathbb{R}^{2n}$ having the measures arising from the original games as marginals; cf. the setting in the optimal mass transport problems. Suitable choices will be helpful in the proofs.

After finishing the paper it has come to our attention that couplings of stochastic processes have been employed in the study of regularity for second order linear uniformly parabolic equations with continuous highest order coefficients, see for example [10], [11], and [21]. The method here has also some similarities to the Ishii–Lions method [6], see also for example [20]. However, we do not rely on the theorem of sums in the theory of viscosity solution, but the proofs are built on the ideas arising from the game theory.

Our motivation to study the above problems is threefold: First, the study of stochastic games has received a lot of attention on their own right because of deep mathematical questions that arise and also due to their central role in many applications. Second, the dynamic programming principle can be interpreted as a discretization of the PDE. Thus, results can also be interpreted as results for the corresponding numerical schemes. Third, by passing to the limit with respect to the step size our results imply regularity results for PDEs: In particular, this gives an alternative proof for the Hölder continuity of the viscosity solutions of the normalized $p(x)$-Laplacian

$$\Delta^N_{p(x)} u =: \Delta u + (p(x) - 2)\Delta^N_\infty u = 0,$$

and the infinity Laplacian

$$\Delta^N_\infty u = |\nabla u|^{-2} \sum_{i,j=1}^{n} u_{ij} u_i u_j = 0$$

where u_i and u_{ij} denote the elements of the gradient and the Hessian, respectively.

There is a powerful connection between the classical linear partial differential equations and probability theory. In the nonlinear case, the connection between the games and Bellman–Isaacs equations was established in the 80s. However, a similar connection between the normalized p-Laplace or ∞-Laplace equations (which are discontinuous operators in the gradient variable) and the tug-of-war games with noise was discovered only rather recently in [18,19]. This connection has later been extended or utilized in several different contexts, see for example [1–4,14–16,22].

Please cite this article in press as: H. Luiro, M. Parviainen, Regularity for nonlinear stochastic games, Ann. I. H. Poincaré – AN (2018), https://doi.org/10.1016/j.anihpc.2017.11.009