ARTICLE IN PRESS
ANIHPC:2885

Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN ••• (••••) •••–•••

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Stationary solutions to the compressible Navier–Stokes system with general boundary conditions

Eduard Feireisl [a,b,*,1], Antonín Novotný [c]

[a] Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, CZ-115 67 Praha 1, Czech Republic
[b] Mathematical Institute, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague, Czech Republic
[c] IMATH, EA 2134, Université du Sud Toulon-Var, BP 20132, 83957 La Garde, France

Received 15 February 2017; received in revised form 14 November 2017; accepted 7 January 2018

Abstract

We consider the stationary compressible Navier–Stokes system supplemented with general inhomogeneous boundary conditions. Assuming the pressure to be given by the standard hard sphere EOS we show existence of weak solutions for arbitrarily large boundary data.

Keywords: Compressible Navier–Stokes system; Stationary solution; Inhomogeneous boundary conditions

1. Introduction

The boundary behavior of fluids influences essentially the motion inside their natural physical domain. Basically all real world applications in fluid mechanics contain boundary conditions as the main factor determining the behavior of the fluid. We consider the problem of identifying the stationary motion of a compressible viscous fluid driven by general in/out flux boundary conditions. Specifically, the mass density $\varrho = \varrho(x)$ and the velocity $\mathbf{u} = \mathbf{u}(x)$ of the fluid satisfy the Navier–Stokes system,

$$\mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \lambda \geq 0, \tag{1.3}$$

in $\Omega \subset R^N$, $N = 2, 3$, where $p = p(\varrho)$ is the barotropic pressure. We consider rather general boundary conditions,

* Corresponding author.
E-mail address: feireisl@math.cas.cz (E. Feireisl).
[1] E.F. acknowledges the support of the project LL1202 in the programme ERC-CZ funded by the Ministry of Education, Youth and Sports of the Czech Republic.

https://doi.org/10.1016/j.anihpc.2018.01.001
0294-1449/© 2018 Published by Elsevier Masson SAS.

Please cite this article in press as: E. Feireisl, A. Novotný, Stationary solutions to the compressible Navier–Stokes system with general boundary conditions, Ann. I. H. Poincaré – AN (2018), https://doi.org/10.1016/j.anihpc.2018.01.001

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B, \ \varrho|_{\Gamma_{\rm in}} = \varrho_B, \tag{1.4}$$

where

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \middle| \ \mathbf{u}_B \cdot \mathbf{n} < 0 \right\}, \ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \middle| \ \mathbf{u}_B \cdot \mathbf{n} > 0 \right\}. \tag{1.5}$$

We concentrate on the inflow/outflow phenomena, we have therefore deliberately omitted the contribution of external forces $\varrho\mathbf{f}$. Nevertheless, all results of this paper remain valid also in the presence of external forces.

Investigation and better insight to the equations in this setting is important for many real world applications. In fact this is a natural and basic abstract setting for flows in pipelines, wind tunnels, turbines to name a few concrete examples. In spite of this fact the problem resists to all attempts of its solution for decades. To the best of our knowledge, this is the first work ever treating this system for large boundary data.

Indeed, the only results available in setting (1.1)–(1.5) are those on existence of strong solutions for small boundary data perturbations of an equilibrium state or of a specific given flow in a particular geometry (as e.g. the Poiseuille flow in a cylinder) in isentropic regime, see e.g. Plotnikov, Ruban, Sokolowski [19], Mucha, Piasecki [14], Piasecki [17], Piasecki and Pokorny [18] among others. The only available results on existence of weak solutions for large flows treat system (1.1)–(1.3) with large external force at the right hand side of equation (1.2) and with homogeneous Dirichlet boundary condition for velocity ($\mathbf{u}|_{\partial\Omega} = 0$) or with Navier slip boundary conditions ($\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega=0}$, $(\mathbb{S}(\nabla_x\mathbf{u})\mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$). For the examples of latter relevant results, see Lions [13], Brezina, Novotny [2], Frehse, Steinhauer, Weigant [9], Plotnikov, Sokolowski [22], Jiang, Zhou [12] among others.

Our goal is to establish the existence of a weak solution $[\varrho, \mathbf{u}]$ to problem (1.1)–(1.5) for general *large* boundary data ϱ_B, $\mathbf{u}_B$. Our approach is based on two physically grounded hypotheses:

- **Molecular hypothesis (hard sphere model).** The specific volume of the fluid is bounded below away from zero. Equivalently, the fluid density cannot exceed a limit value $\overline{\varrho} > 0$. Accordingly, the pressure $p = p(\varrho)$ satisfies

$$\lim_{\varrho \to \overline{\varrho}} p(\varrho) = \infty. \tag{1.6}$$

- **Positive compressibility.** The pressure $p = p(\varrho)$ is a non-decreasing function of the density, more precisely

$$p \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho}), \ p(0) = 0, \ p'(\varrho) \geq 0 \text{ for } \varrho \geq 0. \tag{1.7}$$

The reader may consult Carnahan and Starling [3] for the physical background of hypotheses (1.6), (1.7). Although apparently satisfied by any *real* fluid, condition (1.6) eliminates the more standard equations of state $p(\varrho) = a\varrho^\gamma$ used for the isentropic gases.

Clearly, the fact that the density is *a priori* expected to be confined to a bounded interval $[0, \overline{\varrho}]$ facilitates the analysis. On the other hand the presence of non-zero boundary data makes the analysis more difficult. The proper construction of solutions in this setting is far from being obvious. We use a method relying on a suitable elliptic regularization based on adding artificial diffusion terms to both continuity and momentum equations, along with nonlinear flux-type boundary conditions for density. Note that the more standard approximation based on solving directly the transport equation (1.1), see e.g. Plotnikov, Ruban, Sokolowski [19], [20], [21] is unlikely to work for the large data problems. The proof contains a construction of a suitable extension of the boundary velocity field in the class of functions with positive divergence that may be of independent interest. Compactness of the family of approximate solutions is established by means of compensated compactness combined with the monotone operator theory in the spirit of Lions' work [13].

The paper is organized in the following way. In Section 2, we introduce the concept of weak solution to problem (1.1)–(1.5) and state our main result (Theorem 2.1 and Remark 2.3). Section 3 contains preliminary material concerning extension of vector fields given on $\partial\Omega$. The approximate solutions are constructed in Section 4. After having established the necessary uniform bounds in Section 5, the limit in the family of approximate solutions is performed in Section 7.

2. Main result

In order to avoid additional technicalities, we suppose the $\Omega \subset R^N$, $N = 2, 3$ is a bounded simply connected domain with a boundary of class $C^{2,1}$. The boundary data satisfy

Please cite this article in press as: E. Feireisl, A. Novotný, Stationary solutions to the compressible Navier–Stokes system with general boundary conditions, Ann. I. H. Poincaré – AN (2018), https://doi.org/10.1016/j.anihpc.2018.01.001