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ANIHPC:2889

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ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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# Construction and stability of blowup solutions for a non-variational semilinear parabolic system

# Construction et stabilité de solutions explosives pour un système parabolique sémilinéaire non-variationel

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Received 2 November 2016; accepted 2 January 2018

**Abstract**

We consider the following parabolic system whose nonlinearity has no gradient structure:

$$\begin{cases} \partial_t u = \Delta u + |v|^{p-1}v, & \partial_t v = \mu\Delta v + |u|^{q-1}u, \\ u(\cdot,0) = u_0, & v(\cdot,0) = v_0, \end{cases}$$

in the whole space $\mathbb{R}^N$, where $p, q > 1$ and $\mu > 0$. We show the existence of initial data such that the corresponding solution to this system blows up in finite time $T(u_0, v_0)$ simultaneously in $u$ and $v$ only at one blowup point $a$, according to the following asymptotic dynamics:

$$\begin{cases} u(x,t) \sim \Gamma\left[(T-t)\left(1 + \dfrac{b|x-a|^2}{(T-t)|\log(T-t)|}\right)\right]^{-\frac{(p+1)}{pq-1}}, \\ v(x,t) \sim \gamma\left[(T-t)\left(1 + \dfrac{b|x-a|^2}{(T-t)|\log(T-t)|}\right)\right]^{-\frac{(q+1)}{pq-1}}, \end{cases}$$

with $b = b(p,q,\mu) > 0$ and $(\Gamma, \gamma) = (\Gamma(p,q), \gamma(p,q))$. The construction relies on the reduction of the problem to a finite dimensional one and a topological argument based on the index theory to conclude. Two major difficulties arise in the proof: the linearized operator around the profile is not self-adjoint even in the case $\mu = 1$; and the fact that the case $\mu \neq 1$ breaks any symmetry in the problem. In the last section, through a geometrical interpretation of quantities of blowup parameters whose dimension is equal to the dimension of the finite dimensional problem, we are able to show the stability of these blowup behaviors with respect to perturbations in initial data.

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[1] H. Zaag is supported by the ERC Advanced Grant no. 291214, BLOWDISOL and by the ANR project ANAÉ ref. ANR-13-BS01-0010-03.

https://doi.org/10.1016/j.anihpc.2018.01.003


Please cite this article in press as: T.-E. Ghoul et al., Construction and stability of blowup solutions for a non-variational semilinear parabolic system, Ann. I. H. Poincaré – AN (2018), https://doi.org/10.1016/j.anihpc.2018.01.003

**Résumé**

Nous considérons le système parabolique suivant :

$$\begin{cases} \partial_t u = \Delta u + |v|^{p-1}v, & \partial_t v = \mu \Delta v + |u|^{q-1}u, \\ u(\cdot, 0) = u_0, & v(\cdot, 0) = v_0, \end{cases}$$

dans tout l'espace $\mathbb{R}^N$, où $p, q > 1$ et $\mu > 0$. Nous prouvons l'existence d'une donnée initiale telle que la solution associée explose en temps fini $T(u_0, v_0)$ simultanément en $u$ et $v$, et en un unique point $a$. Plus présicément, nous prouvons que cette solution a le comportement asymptotique suivant :

$$\begin{cases} u(x,t) \sim \Gamma \left[ (T-t) \left( 1 + \dfrac{b|x-a|^2}{(T-t)|\log(T-t)|} \right) \right]^{-\frac{(p+1)}{pq-1}}, \\ v(x,t) \sim \gamma \left[ (T-t) \left( 1 + \dfrac{b|x-a|^2}{(T-t)|\log(T-t)|} \right) \right]^{-\frac{(q+1)}{pq-1}}, \end{cases}$$

avec $b = b(p, q, \mu) > 0$ et $(\Gamma, \gamma) = (\Gamma(p, q), \gamma(p, q))$. La construction de cette solution découle de la réduction du problème à un problème de dimension finie et un argument topologique basé sur la théorie de l'index pour conclure. Deux difficultés majeures se sont posées lors de la preuve :

– d'une part, le linearisé autour du profil n'est pas auto-adjoint, même pas pour $\mu = 1$ ;
– d'autre part, lorsque $\mu \neq 1$, cela brise toute symmétrie dans le problème.

Dans la dernière section, grâce à une interprétation géométrique des paramètres du problème, nous prouvons la stabilité du comportement asymptotique des solutions construites par rapport à des perturbations dans les données initiales.


*MSC:* primary 35K50, 35B40; secondary 35K55, 35K57

*Keywords:* Blowup solution; Blowup profile; Stability; Semilinear parabolic system

## 1. Introduction

In this paper we are concerned with finite time blowup for the semilinear parabolic system:

$$\begin{cases} \partial_t u = \Delta u + |v|^{p-1}v, & \partial_t v = \mu \Delta v + |u|^{q-1}u, \\ u(\cdot, 0) = u_0, & v(\cdot, 0) = v_0, \end{cases} \tag{1.1}$$

in the whole space $\mathbb{R}^N$, where

$$p, q > 1, \quad \mu > 0.$$

The local Cauchy problem for (1.1) can be solved in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. We denote by $T = T(u_0, v_0) \in (0, +\infty]$ the maximal existence time of the classical solution $(u, v)$ of problem (1.1). If $T < +\infty$, then the solution blows up in finite time $T$ in the sense that

$$\lim_{t \to T} (\|u(t)\|_{L^\infty(\mathbb{R}^N)} + \|v(t)\|_{L^\infty(\mathbb{R}^N)}) = +\infty.$$

In that case, $T$ is called the blowup time of the solution. A point $a \in \mathbb{R}^N$ is said to be a blowup point of $(u, v)$ if $(u, v)$ is not locally bounded near $(a, T)$ in the sense that $|u(x_n, t_n)| + |v(x_n, t_n)| \to +\infty$ for some sequence $(x_n, t_n) \to (a, T)$ as $n \to +\infty$. We say that the blowup is *simultaneous* if

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