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Singularity formation of the Yang–Mills Flow

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Abstract

We study singularity structure of Yang–Mills flow in dimensions $n \geq 4$. First we obtain a description of the singular set in terms of concentration for a localized entropy quantity, which leads to an estimate of its Hausdorff dimension. We develop a theory of tangent measures for the flow, which leads to a stratification of the singular set. By a refined blowup analysis we obtain Yang–Mills connections or solitons as blowup limits at any point in the singular set.

Keywords: Yang–Mills; geometric flows; singularity analysis

1. Introduction

Given (M^n, g) a compact Riemannian manifold and $E \to M$ a vector bundle, a one parameter family of connections ∇_t on E is a solution to Yang–Mills flow if

$$\frac{\partial \nabla_t}{\partial t} = -D^*_{\nabla_t} F_{\nabla_t}.$$

This is the negative gradient flow for the Yang–Mills energy, and is a natural tool for investigating its variational structure. Global existence and convergence of the flow in dimensions $n = 2, 3$ was established in [16]. Finite time singularities in dimension $n = 4$ can only occur via energy concentration, as established in [18]. More recently this result has been refined in [5,21] to show concentration of the self-dual and antiself-dual energies. Preliminary investigations into Yang–Mills flow in higher dimensions have been made in [7,15,22].

In this paper we establish structure theorems on the singular set for Yang–Mills flow in dimensions $n \geq 4$. Our results are inspired generally by results of harmonic map flow, specifically [12–14]. The first main result is a weak compactness theorem for solutions to Yang–Mills flow which includes a rough description of the singular set of a sequence of solutions. A similar result for harmonic map flow was established in [12]. Moreover, a related result on

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the singularity formation at infinity for a global solution of Yang–Mills flow was established in [9]. We include a rough statement here, see Theorem 4.1 for the precise statement.

Theorem 1.1. *Fix $n \geq 4$ and let $E \to (M^n, g)$ be a vector bundle over a closed Riemannian manifold. Weak $H^{1,2}$ limits of sequences of smooth solutions to Yang–Mills flow are weak solutions to Yang–Mills flow which are smooth outside of a closed set Σ of locally finite $(n-2)$-dimensional parabolic Hausdorff measure.*

The first key ingredients of the proof are localized entropy monotonicities for the Yang–Mills flow, defined in [9], together with a low-entropy regularity theorem [9]. Fairly general methods allow for the existence of the weak limit claimed in Theorem 1.1, and the entropy monotonicities are the key to showing that the singular set is small enough to ensure that the weak limit is a weak solution to Yang–Mills flow. The arguments are closely related to those appearing in [9,12,19].

The second main result is a stratification of the singular set. This involves investigating tangent measures associated to solutions of Yang–Mills flow. In particular we are able to establish the existence of a density for these measures together with certain parabolic scaling invariance properties. One immediate consequence is that we can apply the general results of [23] to obtain a stratification of the singular set. See §5 for the relevant definitions.

Theorem 1.2. *For $0 \leq k \leq n-2$ let*

$$\Sigma_k := \left\{ z_0 \in \Sigma \mid \dim \left(\Theta^0 \left(\mu^*, \cdot \right) \right) \leq k, \forall \mu^* \in T_{z_0}(\mu) \right\}.$$

Then $\dim_{\mathcal{P}} (\Sigma_k) \leq k$ and Σ_0 is countable.

The third main theorem characterizes the failure of strong convergence in the statement of Theorem 1.1 in terms of the bubbling off of Yang–Mills connections. Again, an analogous result for harmonic maps was established in [12]. The proof requires significant further analysis on tangent measures, leading to the existence of a refined blowup sequence which yields the Yang–Mills connection. We give a rough statement below, see Theorem 6.1 for the precise statement.

Theorem 1.3. *Fix $n \geq 4$ and let $E \to (M^n, g)$ be a vector bundle over a closed Riemannian manifold. A sequence of solutions to Yang–Mills flow converging weakly in $H^{1,2}$ either converges strongly in $H^{1,2}$, and the $(n-2)$-dimensional parabolic Hausdorff measure of Σ vanishes, or it admits a blowup limit which is a Yang–Mills connection on S^4.*

A corollary of these theorems is the existence of a either Yang–Mills connection or Yang–Mills soliton as a blowup limit of arbitrary finite time singularities. For type I singularities the existence of soliton blowup limits was established in [22], following from the entropy monotonicity for Yang–Mills flow demonstrated in [8]. The existence of soliton blowup limits for arbitrary singularities of mean curvature flow was established in [10], relying on the structure theory associated with Brakke's weak solutions. A preliminary investigation into the entropy-stability of Yang–Mills solitons was undertaken in [3] and [11]. Those results now apply to studying arbitrary finite-time singularities of Yang–Mills flow, as all admit singularity models which are either Yang–Mills connections or Yang–Mills solitons.

Corollary 1.4. *Fix $n \geq 4$ and let $E \to (M^n, g)$ be a vector bundle over a closed Riemannian manifold. Let ∇_t a smooth solution to Yang–Mills flow on $[0, T)$ such that $\limsup_{t \to T} \left| F_{\nabla_t} \right|_{C^0} = \infty$. There exist a sequence $\{(x_i, t_i, \lambda_i)\} \subset M \times [0, T) \times [0, \infty)$ such that the corresponding blowup sequence converges modulo gauge transformations to either*

(1) *A Yang–Mills connection on S^4.*
(2) *A Yang–Mills soliton.*

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