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Lyapunov exponents of partially hyperbolic volume-preserving maps with 2-dimensional center bundle [☆]

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Abstract

We consider the set of partially hyperbolic symplectic diffeomorphisms which are accessible, have 2-dimensional center bundle and satisfy some pinching and bunching conditions. In this set, we prove that the non-uniformly hyperbolic maps are C^r open and there exists a C^r open and dense subset of continuity points for the center Lyapunov exponents. We also generalize these results to volume-preserving systems.

MSC: 37D25; 37D30

Keywords: Partially hyperbolic diffeomorphisms; Lyapunov exponents; Non-uniform hyperbolicity

1. Introduction

Lyapunov exponents play a key role in understanding the ergodic behavior of a dynamical system. For this reason, it is important to be able to control how they vary with the dynamics and to avoid zero Lyapunov exponents.

One speaks of *non-uniform hyperbolicity* when all the Lyapunov exponents are different from zero almost everywhere with respect to some preferred invariant measure (for instance, a volume measure). This theory was initiated by Pesin and has many important consequences, most notably: the stable manifold theorem (Pesin [22]), the abundance of periodic points and Smale horseshoes (Katok [16]) and the fact that the fractal dimension of invariant measures is well defined (Ledrappier and Young [18] and Barreira, Pesin and Schmelling [6]).

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In the context of partially hyperbolic volume-preserving systems we study the following classical problems: openness of the set of non-uniformly hyperbolic diffeomorphisms and continuity of the center Lyapunov exponents for the C^r topology with $r \geq 2$.

For the C^1 topology, Mañé [20] observed that an area-preserving diffeomorphism is a continuity point for the Lyapunov exponents only if it is either Anosov or all its Lyapunov exponents are equal to zero almost everywhere. His arguments were completed by Bochi [7] and were extended to arbitrary dimension by Bochi and Viana [8,9]. In particular, Bochi [8] proved that every partially hyperbolic symplectic diffeomorphism can be C^1-approximated by partially hyperbolic diffeomorphisms whose center Lyapunov exponents vanish. This implies that the set of non-uniformly hyperbolic systems is *not* C^1 open. Our first result proves that the situation is different when we consider the C^r topology with $r \geq 2$.

Let $\mathcal{B}^r_\omega(M)$ denote the subset of partially hyperbolic symplectic systems which are accessible, have 2-dimensional center bundle and satisfy some pinching and bunching conditions. (All the keywords will be recalled in the next section.) This set has two important properties: it is a C^1 open set and every $f \in \mathcal{B}^r_\omega(M)$ is ergodic. Moreover, by Theorem A in [26], if $M = \mathbb{T}^{2d}$ with $d \geq 2$, $\mathcal{B}^r_\omega(M)$ is non-empty.

Theorem A. *Let $r \geq 2$. The subset of non-uniformly hyperbolic diffeomorphisms in $\mathcal{B}^r_\omega(M)$ is C^r open.*

The continuity of Lyapunov exponents has been extensively studied for the case of linear cocycles. Theorem C in [7] implies that discontinuity of Lyapunov exponents is typical for continuous $SL(2,\mathbb{R})$-valued cocycles. However, there are some contexts where continuity has been established. Bocker and Viana [10] and Malheiro and Viana [19] proved continuity of Lyapunov exponents for random products of 2-dimensional matrices in the Bernoulli and in the Markov settings. More recently, still for 2-dimensional cocycles, Backes, Brown and Butler [5] proved that continuity of Lyapunov exponents holds in the realm of fiber-bunched Hölder cocycles over any hyperbolic systems with local product structure. In higher dimension, continuity of the Lyapunov exponents for i.i.d. random products of matrices has been announced by Avila, Eskin and Viana [1]. Our second theorem provides a result about continuity of Lyapunov exponents for diffeomorphisms.

Theorem B. *Let $r \geq 2$. There exists a C^r open and dense subset $\mathcal{U} \subset \mathcal{B}^r_\omega(M)$ such that every $g \in \mathcal{U}$ is a continuity point for the center Lyapunov exponents in the C^r topology.*

Moreover, we are able to extend Theorem A and Theorem B for partially hyperbolic volume-preserving systems.

2. Preliminaries and statements

A diffeomorphism $f: M \to M$ of a compact manifold M is *partially hyperbolic* if there exist a nontrivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

invariant under the derivative map Df, a Riemannian metric $\|\cdot\|$ on M, and positive continuous functions χ, $\widehat{\chi}$, ν, $\widehat{\nu}$, γ, $\widehat{\gamma}$ with

$$\chi < \nu < 1 < \widehat{\nu}^{-1} < \widehat{\chi}^{-1} \quad \text{and} \quad \nu < \gamma < \widehat{\gamma}^{-1} < \widehat{\nu}^{-1},$$

such that for any unit vector $v \in T_pM$,

$$\begin{aligned} \chi(p) < \left\|Df_p(v)\right\| &< \nu(p) && \text{if } v \in E^s(p), \\ \gamma(p) < \left\|Df_p(v)\right\| &< \widehat{\gamma}(p)^{-1} && \text{if } v \in E^c(p), \\ \widehat{\nu}(p)^{-1} < \left\|Df_p(v)\right\| &< \widehat{\chi}(p)^{-1} && \text{if } v \in E^u(p). \end{aligned} \tag{1}$$

Partial hyperbolicity is a C^1 open condition, that is, any diffeomorphism sufficiently C^1-close to a partially hyperbolic diffeomorphism is itself partially hyperbolic. Moreover, if $f: M \to M$ is partially hyperbolic, then the stable